CSCanada
Studies in Mathematical Sciences
Vol. 7, No. 2, 2013, pp. 1-24
DOI:10.3968/j.sms.1923845220130702.2866

ISSN 1923-8444 [Print]
ISSN 1923-8452 [Online]
www.cscanada.net
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Stability of Nested Queue Model With Finite Waiting Capacity

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Received: August 20, 2013/ Accepted: November 14, 2013/ Published: November 30, 2013

Abstract: The paper describes a mathematical model to identify the conditions of stability for a two stage nested queue model. We envisioned the two stages as two stations in a hospital that a patient passes through before completing service. In the first station there are two servers (i.e., nurses) which attend to the patient. When a patient arrives to the first station he/she is served by one of the two servers, the server serves the patient and stays with the customer until the service in the second station of the model is completed. In the second station, another server (i.e., doctor) attends to the customer with the cooperation of first station server. The service rates of the servers and arrival rates of the patients follow exponential distribution. The mathematical model is also used to determine the conditions that allow the patients in system to be served effectively. To estimate the conditions of stability, the steady state probabilities have been calculated and stability behavior has been given for special cases.

Key words: Nested queue; Steady state; Probability; Stability

Bhatt, A., & Jang, J. (2013). Stability of Nested Queue Model With Finite Waiting Capacity. *Studies in Mathematical Sciences, 7*(2), 1-24. Available from URL: http://www.cscanada.net/index.php/sms/article/view/j.sms.1923845220130702.2866 DOI: http://dx.doi.org/10.3968/j.sms.1923845220130702.2866

1. INTRODUCTION

"A nested queueing system is a hierarchical system of multiple queues such that service demand in one queue is created by a subset of the units in the system of another queue higher in the hierarchy of possibly providing an entirely different service" (Modi, 1974, p.220). A nested queue can be found in many manufacturing and service systems, including a GI (Gastrointestinal) unit of a hospital which motivated this research. The study determines the best conditions for the nested queue system to provide service for the customers or we have called this effective service. The conditions derived in this analysis give information to policy makers in a hospital setting to determine.

a. The maximum number of patients that a nested queue model can service with pre-determined arrival rate of patients and service rates of the servers.

b. The best allocation of arrival rates for patients in a nested queue system where the policy makers have determined a fixed capacity of patients and pre-determined service rates of the servers.

c. The best services rates for the servers, given pre-determined system capacity and arrival rates of the patients.

The best conditions of service is achieved by determining the stationary state probabilities, ($P_{(i,j)}$), in the model for different parameters of the system. When the values of $P_{(i,j)}$ are found to be greater than zero the nested queue system is considered stable and we can confirm the best conditions for service. When $P_{(i,j)}$ is equal to zero we say the nested queue system is unstable and the conditions of service to the patients should not be considered.

The stability analysis can be used by policy makers for purposes of designing nested queue systems where the importance is crowd control and efficient usage of resources.

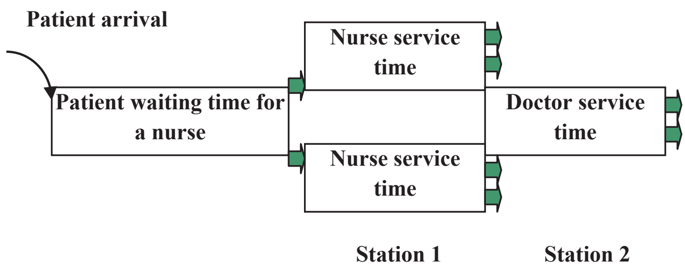

Figure 1
Diagram Depicting the Flow of the Patients Through the Two Stations

The nested queue system has two stations that a patient passes through to complete the service. The first station has two servers (nurses) while the second station consists of one server (doctor).

Flow of the patient is as follows:

a. When a patient arrives at the first station, he/she is served by one of the two nurses.

b. After completing the service in the first station the patient moves to the second station accompanied by the nurse. In the second station the doctor attends to the patient with the help of the nurse (therefore the nurse and doctor are both busy at the second station).

c. In the meantime if another patient arrives to the system he/she will receive service by the second nurse. If service is completed first by the second nurse in the first station before the service is completed by the nurse and doctor at the second station, the nurse and the patient in the first station do not proceed to the second station for further service. At this instant the queueing system is blocked. When the queueing system is blocked no other can get new service and the patients have to wait. Figure 1 depicts the flow of the patient through the two stations. In this queueing system the service of the nurse is nested into the service of the doctor.

The nested queue model was first described by Modi (1974) who analyzed the air traffic control (ATC) sector as a nested queue model. Modi applied the model for the interaction between the airplane traffic and the control room communications. Modi's model consisted of two stages, in which the first stage consisted of multiple servers in parallel with the second stage having one server. The first stage of Modi's model consisted of S number of parallel servers receiving messages about transitting of planes in an ATC Sector. The second stage of the Modi's nested queue model consisted of a server that provids messages to the aircraft giving information on clearance for the aircraft and course guidance to move through the ATC sector. Modi's case dealt with the service demand of the second stage queue system being created by a subset of the units in the first queue system. If the first queue system had reached capacity, the second queue system still served but the planes were not allowed into the first system and would need to wait. The paper studies how the operating characteristics of the second system changes with the parameters of the first. This nested queue model has blocking.

The concept of blocking in Queueing Networks has been studied extensively by Perros (1994). Perros and Foster (1980) studied the blocking process for two queuing systems, a two station and a three station queueing networks. For the two station queue systems the analysis was done for two model types. The first model was a two station queue network where each station had a single server, and the second was a two station queue network where the first station consisted of "n" number of symmetrical servers in parallel and one server in the second station. In the three station network model, the first station had "n" symmetrical servers, the second station had two servers and the third station had one server. Perros and Foster (1980) have considered a model where the first station server is saturated (the server in the first station becomes saturated, when working 100% of the time) and the capacity of the waiting area is infinite.

The nested queue model is similar to the Perros and Foster model. Just like the Perros and Foster model blocking is also considered in the nested queue model.

Grassmann and Tavakoli (2005) have also described with a system with moving servers. They have defined a two station model which has one server in each station. The first station has a queue with infinite capacity while the second station has a finite capacity. The model has been studied under the assumption that should a server be idle, he/she can move to the other busy station and work with the server in that station (allowing the servers to cooperate and work together). In this model if server 1 is idle it helps server 2 to attend to the customers, which in turn increases the service rate of server 2 at the second station. Similarly if server 2 is idle, it helps server 1, which then increases the rate at the first service station. The same concept of cooperation of servers has also seen in the nested queue model presented in this paper.

2. FORMULATION OF THE PROBLEM

2.1 Model Description

The goal of the analysis is to find the condition of stability for the model as described above in Figure 1. Figure 2 depicts the two station nested queue model, where the first station has two servers and the second station has only one server. System has the capacity of K patients. At time t, the patient V arrives to the facility at B with finite capacity (M) (M includes the number of patients waiting to receive service at B and also receiving service at C). Patient A finds two nurses (N1 and N2) at service area (C). Once the patient is attended by one nurse, N1, they both move to station 2. In the second station, if the doctor (DC) is available, patient A receives service immediately. While patient A is with the doctor (DC) (at service area (D)), the nurse N1 is still attending to the patient A.

Consider the next patient A_e at time t_e, arrives at the facility B where the nurse N2 attends to the patient. Once the service is completed at area C, the nurse N2 and patient A_e would move to area D. As the doctor is busy with patient A, nurse N2 and patient A_e have to wait at the area C. When nurse N2 is waiting in area C, the system is blocked (when a system is blocked no other patient is served and any new patients have to wait). In the second station only one patient can be served at a time. Let N be the capacity of the second station.

In this model once system capacity K (which equal to M+N) is reached, no other patients are allowed into the system.

The arrival of the customers to the first station is assumed to be Poisson distribution (of rate λ) and service times in each station are exponentially distributed with the rates μ_1 and μ_2 respectively. The customers are served on first come first basis. Since let $P_{(m,n)}$ be the steady state probability of the system in the state (m,n). Let $P^*_{(m,n)}$ be the steady state probability when then the system is blocked.

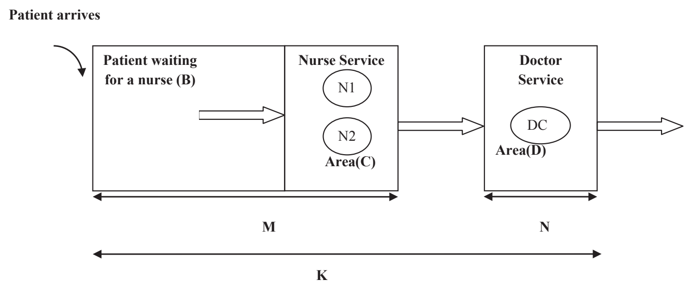

Figure 2
Diagram Depicting the Flow of the Customers Through the Nested System

2.2 Methodology

For this paper the methodology used to analyze the nested queue model is very similar to what was used by Hunt (1956). Hunt examined a finite buffer between two stations in a Tandem Queue model. He used a single server with a sequential two-station model to compare traffic intensities for the three basic cases, namely, (i) where an infinite buffer exists between the stations, with the exception that the first station may have an finite buffer; (ii) where zero buffer exists between the stations and (iii) where a finite buffer (greater than zero (0)) exists between the stations, with the exception that the first station may have an infinite buffer. A similarity between the Hunt case and the nested queue model is that both have zero buffers between the stations. Hunt's methodology was also used by Perros (1994) for solving tandem queue problems with the blocking principle.

Now we first show the changes of system states upon arrival or departure of a customer. The state transition diagram is given in Figure 3.

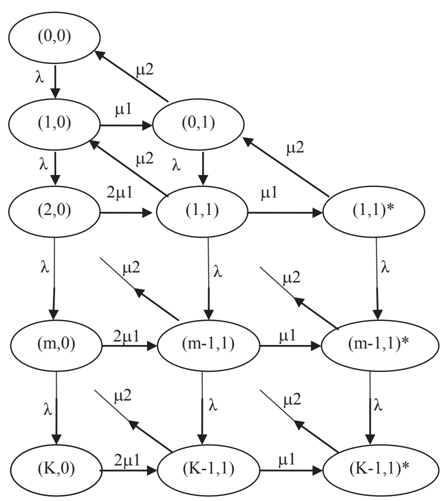

Figure 3
Transition Diagram Depicting the Effect of Arrivals to the Sequential Nested System

The diagram starts with the state (0, 0) (meaning no patient exists). When a patient arrives to the system, the value of m =becomes 1, hence state changes to (1,0). Once service is received by the patient in the first station, the patient moves to the second station and receives service, the state becomes (0,1). Here if the service is completed in the second station the patient leaves the system and state returns to (0,0). In general, our system is assumed to begin with (m,n) when we have an arrival in the system, the state changes from (m,n) to (m+1,n). When patient moves from station 1 to station 2, the state changes to (m-1, n+1). After service is completed at the station 2 then state of the system changes from (m-1, n+1) to (m-1, n). If there is an arrival during the process then the value of (m-1) can change to (m). If the system is blocked, the state is defined by (m-1, n+1)*. The symbol in the transition diagram (*) denotes blocking. We continue to analyze the system until we have reached system capacity denoted by (K,n).

Using Figure 3, the governing steady state equations are obtained. For the nested queue model we are interested in the stationary state probabilities $P_{(m,n)}$ or $P_{(i,j)}$ which can be obtained by solving the system of steady state equations.

2.3 Analysis

2.3.1 Analysis of System Capacity for K = 2

We first consider capacity K = 2. The transition diagram becomes as follows.

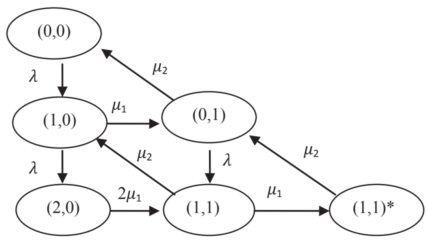

Figure 4
Transition Diagram Depicting the Effect of Arrivals to the Sequential Nested System for System Capacity K=2.

The steady equations are given as follows:

$$\lambda P_{(0,0)} = \mu_2 P_{(0,1)} \tag{1}$$

$$(\lambda+\mu_2)P_{(0,1)} = \mu_2 P*_{(1,1)} + \mu_1 P_{(1,0)} \tag{2}$$

$$(\lambda+\mu_1)P_{(1,0)} = \lambda P_{(0,0)} + \mu_2 P_{(1,1)} \tag{3}$$

$$(\mu_1+\mu_2)P_{(1,1)} = \lambda P_{(0,1)} + 2\mu_1 P_{(2,0)} \tag{4}$$

$$(\mu_2)P*_{(1,1)} = \mu_1 P_{(1,1)} \tag{5}$$

$$(2\mu_1)P_{(2,0)} = \lambda P_{(1,0)} \tag{6}$$

We need this sum of probability to be unity:

$$P_{(0,0)} + P_{(0,1)} + P_{(1,0)} + P_{(1,1)} + P*_{(1,1)} = 1 \tag{7}$$

Solving for $P_{(0,0)}$ we get

$$P_{(0,0)} = \frac{2{\mu_1}^2{\mu_2}^2}{2{\mu_1}^2{\mu_2}^2 + 2\lambda\mu_1\mu_2(\mu_1+\mu_2) + \lambda^2(2\mu_1\mu_2 + 2{\mu_1}^2 + {\mu_2}^2)} \tag{8}$$

We know that

$$P_{(0,1)} = \frac{\lambda}{\mu_2} P_{(0,0)} = \frac{2\lambda{\mu_1}^2\mu_2}{2{\mu_1}^2{\mu_2}^2 + 2\lambda\mu_1\mu_2(\mu_1+\mu_2) + \lambda^2(2\mu_1\mu_2 + 2{\mu_1}^2 + {\mu_2}^2)} \quad (9)$$

$$P_{(1,1)} = \left(\frac{\lambda^2}{\mu_2\mu_1}\right) P_{(0,0)} = \frac{2\lambda^2\mu_1\mu_2}{2{\mu_1}^2{\mu_2}^2 + 2\lambda\mu_1\mu_2(\mu_1+\mu_2) + \lambda^2(2\mu_1\mu_2 + 2{\mu_1}^2 + {\mu_2}^2)} \quad (10)$$

$$P*_{(11)} = \frac{\lambda^2}{{\mu_2}^2} P_{(00)} = \frac{2\lambda^2{\mu_1}^2}{2{\mu_1}^2{\mu_2}^2 + 2\lambda\mu_1\mu_2(\mu_1+\mu_2) + \lambda^2(2\mu_1\mu_2 + 2{\mu_1}^2 + {\mu_2}^2)} \quad (11)$$

$$P_{(1,0)} = \frac{\lambda}{\mu_1} P_{(0,0)} = \frac{2\lambda\mu_1{\mu_2}^2}{2{\mu_1}^2{\mu_2}^2 + 2\lambda\mu_1\mu_2(\mu_1+\mu_2) + \lambda^2(2\mu_1\mu_2 + 2{\mu_1}^2 + {\mu_2}^2)} \quad (12)$$

$$P_{(2,0)} = \frac{\lambda^2}{2{\mu_1}^2} P_{(00)} = \frac{\lambda^2{\mu_2}^2}{2{\mu_1}^2{\mu_2}^2 + 2\lambda\mu_1\mu_2(\mu_1+\mu_2) + \lambda^2(2\mu_1\mu_2 + 2{\mu_1}^2 + {\mu_2}^2)} \quad (13)$$

2.3.2 Analysis of System Capacity for K = 3

Next we look at capacity of K=3. The transition diagram becomes as follows:

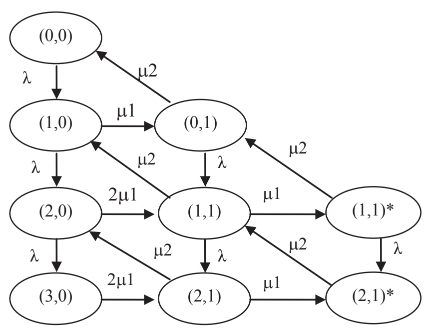

Figure 5
Transition Diagram Depicting the Effect of Arrivals to the Sequential Nested System Where System Capacity (K) = 3

The steady state equations are as follows:

$$0 = -\lambda P_{(0,0)} + \mu_2 P_{(0,1)} \quad (14)$$

$$0 = -(\lambda+\mu_2)P_{(0,1)} + \mu_2 P*_{(1,1)} + \mu_1 P_{(1,0)} \quad (15)$$

$$0 = -(\lambda+\mu_1)P_{(10)} + \lambda P_{(00)} + \mu_2 P_{(11)} \quad (16)$$

$$0 = -(\lambda+\mu_2)P*_{(11)} + \mu_1 P_{(11)} \quad (17)$$

$$0 = -(\lambda+\mu_1+\mu_2)P_{(1,1)} + 2\mu_1 P_{(2,0)} + \lambda P_{(0,1)} + \mu_2 P*_{(2,1)} \quad (18)$$

$$0 = -(\lambda+2\mu_1)P_{(2,0)} + \lambda P_{(1,0)} + \mu_2 P_{(2,1)} \quad (19)$$

$$0 = -(\mu_2)P*_{(2,1)} + \mu_1 P_{(2,1)} + \lambda P*_{(1,1)} \quad (20)$$

$$0 = -(\mu_1+\mu_2)P_{(2,1)} + 2\mu_1 P_{(3,0)} + \lambda P_{(1,1)} \quad (21)$$

$$0 = -(2\mu_1)P_{(3,0)} + \lambda P_{(2,0)} \quad (22)$$

The sum of all the probabilities

$$P_{(0,0)} + P_{(0,1)} + P_{(1,0)} + P_{(1,1)} + P*_{(1,1)} + P_{(2,0)} + P_{(2,1)} + P*_{(2,1)} + P_{(3,0)} = 1 \quad (23)$$

Solving the above equations we get $P_{(0,0)}$

$$P_{(0,0)} = \frac{4\lambda^2\mu_1{}^3\mu_2{}^3 + 10\lambda\mu_1{}^4\mu_2{}^3 + 10\lambda\mu_1{}^3\mu_2{}^4 + 4\mu_1{}^5\mu_2{}^3 + 8\mu_1{}^4\mu_2{}^4 + 4\mu_1{}^3\mu_2{}^5}{\begin{array}{c}2\lambda^6\mu_1{}^2 + 2\lambda^6\mu_1\mu_2 + \lambda^6\mu_2{}^2 + 8\lambda^5\mu_1{}^3 + 14\lambda^5\mu_1{}^2\mu_2 + 10\lambda^5\mu_1\mu_2{}^2 + 3\lambda^5\mu_2{}^3 + 10\lambda^4\mu_1{}^4 + 28\lambda^4\mu_1{}^3\mu_2 \\ +30\lambda^4\mu_1{}^2\mu_2{}^2 + 15\lambda^4\mu_1\mu_2{}^3 + 3\lambda^4\mu_2{}^4 + 4\lambda^3\mu_1{}^5 + 22\lambda^3\mu_1{}^4\mu_2 + 42\lambda^3\mu_1{}^3\mu_2{}^2 + 33\lambda^3\mu_1{}^2\mu_2{}^3 + 10\lambda^3\mu_1\mu_2{}^4 \\ +\lambda^3\mu_2{}^5 + 4\lambda^2\mu_1{}^5\mu_2 + 22\lambda^2\mu_1{}^4\mu_2{}^2 + 38\lambda^2\mu_1{}^3\mu_2{}^3 + 18\lambda^2\mu_1{}^2\mu_2{}^4 + 2\lambda^2\mu_1\mu_2{}^5 + 4\lambda\mu_1{}^5\mu_2{}^2 \\ 22\lambda\mu_1{}^4\mu_2{}^3 + 22\lambda\mu_1{}^3\mu_2{}^4 + 4\lambda\mu_1{}^2\mu_2{}^5 + 4\mu_1{}^5\mu_2{}^3 + 8\mu_1{}^4\mu_2{}^4 + 4\mu_1{}^3\mu_2{}^5\end{array}} \quad (24)$$

Substituting P(0,0) into the other equations we can find the other probabilities, Appendix 4 gives more detail.

As we see in Figure 2 for K=2 there is no occurrence of blocking but for K=3 there is blocking. When we observe the pattern from Figure 3 we see a general pattern emerges. This general pattern of equations can be applied for values of K ≥ 3. For K=2, equation (5) cannot be derived from the general case, by using the general format we obtain

$$0 = -(\mu_2)P*_{(1,1)} + \mu_1 P_{(1,1)} + \lambda P*_{(0,1)} \quad (25)$$

which is not same as (5), since $\lambda P*_{(0,1)}$ does not exist. Hence the steady state equations for K=2 have been considered separately.

2.3.3 Analysis for General Value of K

For general values of K the steady state equations

$$0 = -\lambda P_{(0,0)} + \mu_2 P_{(0,1)} \quad (26)$$

$$0 = -(\lambda+\mu_2)P_{(0,1)} + \mu_2 P*_{(1,1)} + \mu_1 P_{(1,0)} \quad (27)$$

$$0 = -(\lambda+\mu_1)P_{(1,0)} + \lambda P_{(0,0)} + \mu_2 P_{(1,1)} \quad (28)$$

$$0 = -(\lambda+\mu_2)P*_{(1,1)} + \mu_1 P_{(1,1)} \quad (29)$$

$$0 = -(\lambda+2\mu_1)P_{(m,n)} + \lambda P_{(m-1,n)} + \mu_2 P_{(m,n+1)}\ ;\ 2 \le m < K, \quad n = 0 \quad (30)$$

$$0 = -(\lambda+\mu_1+\mu_2)P_{(m,n)} + 2\mu_1 P_{(m+1,n-1)} + \lambda P_{(m-1,n)} + \mu_2 P*_{(m+1,n)}\ ;\ 1 \le m < K-1\ ,\ n = 1 \quad (31)$$

$$0 = -(\lambda+\mu_2)P*_{(m,n)} + \mu_1 P_{(m,n)} + \lambda P*_{(m-1,n)}\ ;\ 2 \le m < K-1,\ n = 1 \quad (32)$$

$$0 = -(2\mu_1)P_{(m,n)} + \lambda P_{(m-1,n)}\ ;\ m = K,\ n = 0 \quad (33)$$

$$0 = -(\mu_1+\mu_2)P_{(m,n)} + \lambda P_{(m-1,n)} + 2\mu_1 P_{(m+1,n-1)}\ ;\ m = K-1,\ n = 1 \quad (34)$$

$$0 = -(\mu_2)P*_{(m,n)} + \mu_1 P_{(m,n)} + \lambda P*_{(m-1,n)}\ ;\ m = K-1, n = 1 \quad (35)$$

The total number of equations is 3K where K= system capacity.

The normalization equation for the probabilities is

$$\sum_{i=0}^{K}\left[P_{(i,0)}\right] + \sum_{i=0}^{K-1}\left[P_{(i,1)}\right] + \sum_{i=1}^{K-1}\left[P^*_{(i,1)}\right] = 1 \quad (36)$$

When we include the normalization equation, the number of equations now total to 3K+1.

Following Konheim and Reiser (1976) the system is considered to be stable if

$P_{(i,j)} > 0$, $\frac{\lambda}{2\mu_1} < 1$ and $\frac{\lambda}{\mu_2} < 1$ for all i and j and unstable if $P_{(i,j)} = 0$ for any i and j.

When the system is stable, the queue length can be obtained as

$$L = \sum_{i=1}^{K} iP_{(i,0)} + \sum_{i=1}^{K-1} i\left[P_{(i1)} + P^{*}{}_{(i,1)}\right] \quad (37)$$

3. NUMERICAL ANALYSIS

Using the above equations, a stability analysis was done for the nested queue model to determine the best conditions of service for the customers in the nested queue model. This analysis provides information on how a nested queue system can be managed for fixed and variable values of arrival rates, service rates and system capacities.

a. To determine the system capacity for a nested queue model given pre-determined service rates for each station (μ_1 and μ_2) and arrival rates of the patients.

b. To determine the best arrival rate of the patients given a pre-determined system capacity and service rates for each station (μ_1 and μ_2).

c. To determine the best service rate for the server in the first station(μ_1) given a pre-determined system capacity, arrival rate for patients and service rate for the second station (μ_2).

d. To determine the service rate of the server in the second station(μ_2) given a pre-determined system capacity, arrival rate for patients and service rates for the first station (μ_1).

For these cases we have summarized the results in Table 1 and then discussed each case in detail. The table depicts the system capacities where the nested queue model is stable and unstable, as well as the bifurcation point of stability.

Table 1
Results of Stability Analysis for the Considered Cases

Case	Fixed values	Stable	Unstable	Bifurcation point
1	λ=1, μ_1=3, μ_2=2	K=[2,14]	K=[15,∞]	K=14
2	K=15, μ_1=3, μ_2=2	λ=[0.6,∞]	λ=[0.1,0.5]	λ=0.5
3	K=12, λ=1, μ_2=2	μ_1=[0,11.4]	μ_1=[11.5, ∞]	μ_1=11.4
4	K=12, λ=1, μ_1=3	μ_2=[0,4.4]	μ_2=[4.5, ∞]	μ_2=4.4

The table depicts the system parameters where the nested queue model is stable, unstable and the bifurcation point of stability.

3.1 Stability Analysis for Fixed Arrival Rate and Service Rates

For the first aim, we fixed the arrival rates and service rates for the nested queue model and determined at what system capacity the servers in the nested queue model can provide the best service to the patients. These best conditions of service for the pre-determined parameters of the model we have termed as effective service. The effective service is determined by the stability region. By calculating the steady state probabilities we can determine when the system remains stable ($P_{(m,n)}$ is greater than zero).

For the present model, we chose λ= 1, μ_1=3 and μ_2=2 for various values of K (3,4,....) and have observed under what system capacity the system remains stable. We saw that for K ε [2,14] the system was stable and unstable for K ε [15,∞], meaning the probabilities of for K ε [2,14] are positive. These results have been summarized in Table 1. Figure 6 below shows the behavior of probabilities $P_{(12,0)}$, $P_{(14,0)}$ and $P_{(15,0)}$.

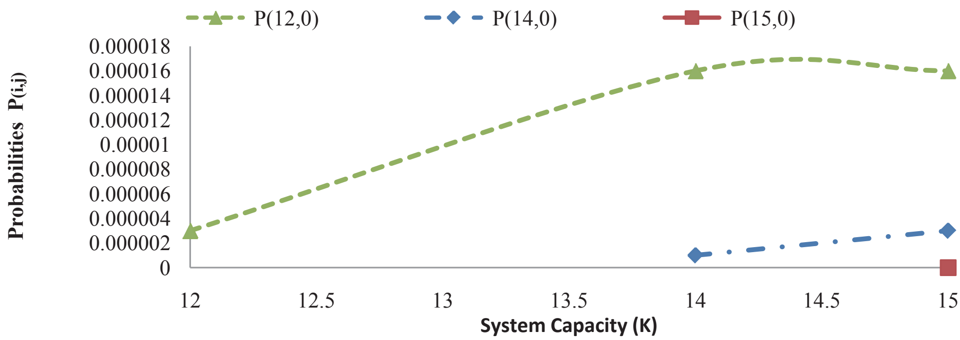

Figure 6
Stability Behavior of Probabilities for λ= 1, μ_1=3 and μ_2=2

The graph shows bifurcation point (where the system changes its behavior from stability to instability, meaning $P_{(m,n)}$ changes from being greater to zero to zero). We take the last/first value of the parameter where the system is stable at K=14. From Table 1-AC (Appendix C) we see that at K=14, $P_{(14,0)}$ =0.000001, which is close to zero but at K=15, $P_{(15,0)}$ = 0 . The results means that for the given arrival and service rates, the maximum capacity of the system is K=14, meaning the system capacity where satisfactory service is provided to the patients. For this model the average queue length for the K=14 is L= 0.953859.

3.2 Stability Analysis for Fixed System Capacity and Service Rates

Next we fixed the system capacity, the service rates for the two stations and made the arrival rates variable. We have calculated the values for $P_{(i,j)}$ for K=15, μ_1=3 and μ_2=2. The arrival rates of the patient are varied, starting from λ = 0.1 until the system becomes stable.

As shown in Table 1 we have observed that the system remains unstable for λ ε (0, 0.5] and stable for λ ε [0.6, ∞). Figure 7 below displays the stability region for probabilities $P_{(9,0)}$, $P_{(12,0)}$ and $P_{(15,0)}$.

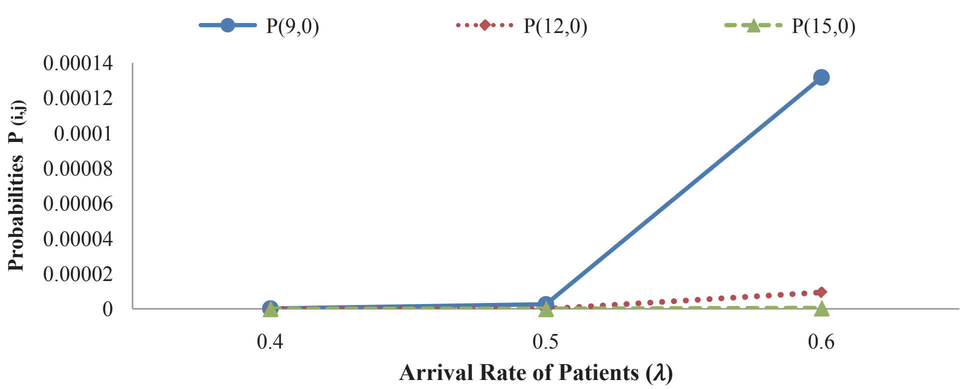

Figure 7
Stability Behavior of Probabilities for K = 15, μ_1=1 and μ_2=2

In Table 1-AC (Appendix C), it can be seen that at λ = 0.4 the probabilities start at slightly above 0 ($P_{(9,0)}$ = 0.0000002) and then move to zero, while at λ = 0.5 the system is again tends to instability ($P_{(9,0)}$ = 0.00000026, $P_{(12,0)}$= 0). At λ = 0.6 the values are positive, at $P_{(15,0)}$ = 0.0000004, the value is just above 0, hence giving the minimum value of arrival rate. The graph shows the bifurcation point is λ=0.6. This means that the scheduled arrivals of patient should be at least 0.6/hr for an effective service. For fixed capacity and service rates we have to adjust the arrival rate in order to give an effective service. The average queue length for the stable system is L= 1.767123.

3.3 Stability Analysis When System Capacity, Service Rate for 2nd Station Server and Arrival Rates Are Kept Constant and Service Rate for 1st Station Server Is Variable

Similar to above, we fixed K = 12, λ =1 and μ_2=2. The service rate μ_1 is varied until the system is unstable. It is interesting to note that the system is stable for μ_1 ε (0, 11.4] and unstable for μ_1 ε [11.5, ∞). In Figure 8, the stability regions are shown for $P_{(5, 0)}$, $P_{(9, 0)}$ and $P_{(12, 0)}$.

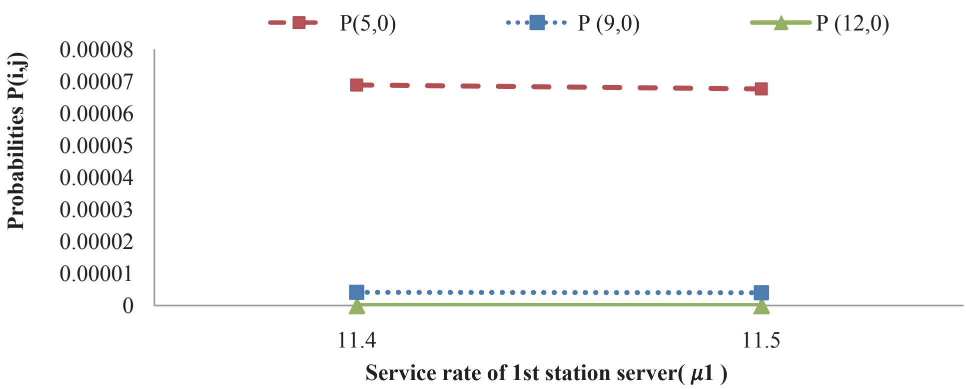

Figure 8
Stability Behavior of Probabilities for K = 12, λ= 1 and μ_2=2

We see that at μ_1=11.4, $P_{(5,0)}$=0.0000647, $P_{(9,0)}$= 0.0000041 and $P_{(12,0)}$=0.00000001, hence the system is stable. In Table 2-AC (Appendix C) we see that at μ_1=11.5, $P_{(5, 0)}$ =0.0000636, $P_{(9,0)}$ = 0.000004 and $P_{(12, 0)}$ = 0, therefore the system is unstable. This means that for K= 12 and for the given arrival rate and service rate for the second station server the service rate of the first sever should be adjusted (e.g. the maximum value of μ_1 < 11.5) in order to provide a satisfactory service to the patients. The average queue length for the stable system is L= 0.2756027.

3.4 Stability Analysis When System Capacity, Service Rate for 1st Station Server and Arrival Rates Are Kept Constant and Service Rate for 2nd Station Server Is Variable

For this case we have fixed K = 12, λ =1 and μ_1=3 and varied μ_2. In Figure 9, the stability regions are shown for $P_{(80)}$ and $P_{(12, 0)}$.

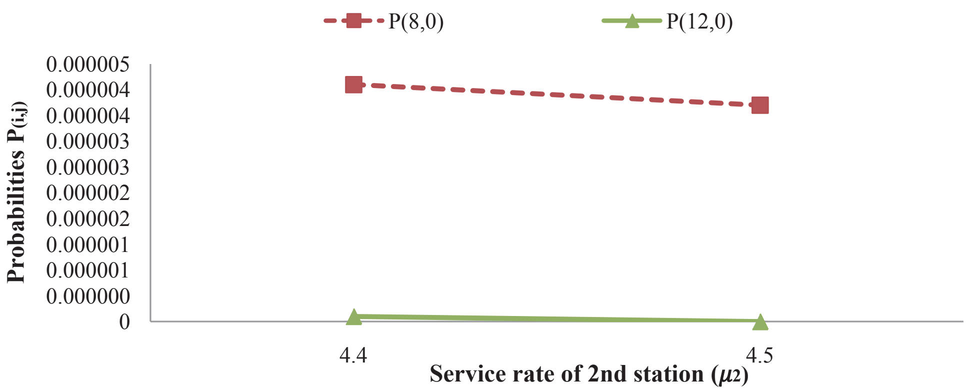

Figure 9
Stability Behavior of Probabilities for K = 12, λ= 1 and μ_1=3

We found that at μ_2 = 4.5, $P_{(8, 0)}$ = 0.0000042 and $P_{(12, 0)}$ = 0 hence the system is unstable. For μ_2 = 4.4, $P_{(8, 0)}$ = 0.0000042 and $P_{(12, 0)}$ = 0.00000001 the system is stable. The bifurcation point as seen on the graph is at $P_{(12, 0)}$ when μ_2 = 4.5. Table 1 shows the summary of the results for the case. From the results in Table 2-AC (Appendix C), for K= 12 and the given arrival rate and service rate of the first station server, the service rate of the second station server should be adjusted (e.g. $\mu_2 \leq 4.4$) to get the effective service. The average queue length for the stable system is L = 0.6602436.

Further analysis was done where the nested queue system was analyzed under variable service rates and arrival rates to see the effect on the utilization of the servers as well as overall the system behavior. The system analyzed was for a stable system. At K=3, we see that the system is stable hence we observed the behavior at this system capacity. Table 2 depicts the results of the system under variable service and arrival rates.

Table 2
Results Depicting the Effect of Variable λ on the Steady State Probabilities, Average Queue Lengths and Utilization of Servers of System at K = 3, where μ_1 = 5 and μ_2= 4

	λ= 1,	λ= 1.2,	λ= 1.32,	λ= 1.45,	λ= 1.6,	λ= 1.76,	λ= 1.936,	λ= 2.1296
P(0,0)	0.615	0.5535	0.5191	0.4839	0.4459	0.4084	0.3706	0.333
P(0,1)	0.1538	0.1661	0.1713	0.1754	.01784	0.1797	0.1794	0.1773
P(1,0)	0.1258	0.137	0.1422	0.1465	0.1502	0.1527	0.154	0.154
P(1,1)	0.0349	0.0464	0.0533	0.0609	0.0694	0.0783	0.0876	0.0971
P*(1,1)	0.0349	0.0446	0.0501	0.0558	0.062	0.068	0.0736	0.0792
P(2,0)	0.0134	0.0177	0.0204	0.0233	0.0288	0.0301	0.0337	0.0375
P(2,1)	0.0054	0.0084	0.0108	0.0136	0.0171	0.0212	0.0261	0.0319
P*(2,1)	0.0154	0.0241	0.0301	0.0372	0.0461	0.0564	0.0683	0.082

To be continued

Continued

	λ= 1,	λ= 1.2,	λ= 1.32,	λ= 1.45,	λ= 1.6,	λ= 1.76,	λ= 1.936,	λ= 2.1296
P(3,0)	0.0013	0.0021	0.0027	0.0034	0.0043	0.0053	0.0065	0.008
Stable/unstable	Stable	Stable	Stable	Stable	Stable	Stable	Stable	Stable
Average length of queue in first stage	0.0221	0.0346	0.0436	0.0542	0.0675	0.0829	0.1009	0.1219
Average length of queue in system	0.315065	0.34555	0.359165	0.369942	0.379684	0.381405	0.380347	0.374492
Utilization of nurse	0.1	0.12	0.132	0.145	0.16	0.176	0.1936	0.212696
Utilization of doctor	0.239925	0.2841	0.309903	0.3373425	0.36836	0.400752	0.43565	0.472664

In Table 2 we see that as we increase the arrival rate of the patients the length of queue increases for fixed values of service rates with respect to servers in both stations. With the increase in arrival the utilization of nurses and doctors also increase. We see that when we the arrival rate of the patients increase from 80% to 90% from the initial arrival rate, the increase in utilization of nurse and doctor increased by 10 % and 9% respectively. From this we see the increase in arrival rate causes the nurses are busier. When the increase in arrival rate is 50% to 60% the increase in utilization of nurse and doctor is again seen to increase by 10% and 9%.

We looked at the effect on the average length of queue at the first station with change in arrival rate of the patient. These results are seen in Figure 10.

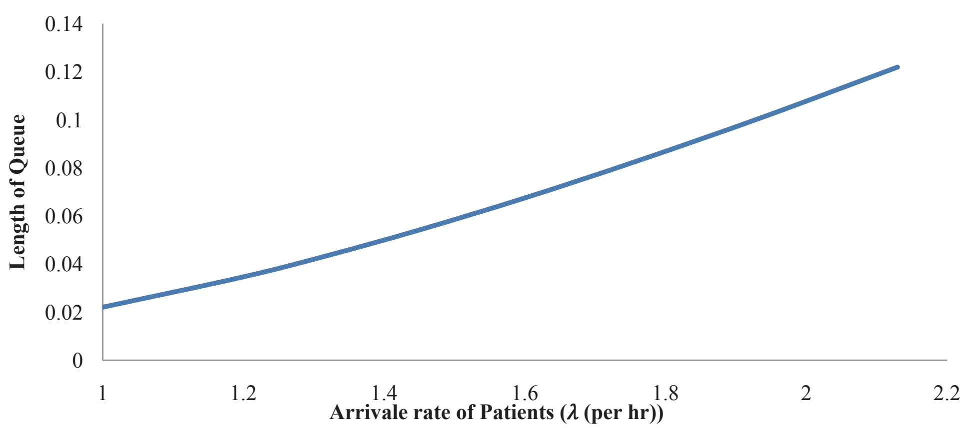

Figure 10
Graph Shows the Effect on the Average Length of Queue for a Stable System with Change in Inter-Arrival Rate of the Patient with Fixed K, μ_1, μ_2 and Variable λ

We see that when the service rate for the second station is fixed and the arrival rate of the patients is fixed, as the service rate of the first station servers changes the length in queue decreases, as the nurse serves patients faster in the first station the stations are now able to move to the second station, hence the length of queue in the first stage decreases with increase in movement of patients.

Further analysis was done to see the effect of variable service rate of the first station servers on the steady state probabilities, average queue lengths and utilization of servers of system at K = 3, where λ and μ_2 are fixed. The results are seen in Table 3.

Table 3
Results Depicting the Effect of Variable μ_1 on the Steady State Probabilities, Average Queue Lengths and Utilization of Servers of System at K = 3, where λ =1 and μ_2 =4

	μ_1=5	μ_1=6	μ_1=6.6	μ_1=7.26	μ_1=7.986	μ_1=8.785	μ_1=9.663	μ_1=10.63
P(0,0)	0.615	0.6364	0.6463	0.6559	0.6638	0.6716	0.6786	0.682
P(0,1)	0.1538	0.1591	0.1616	0.164	0.166	0.1679	0.1697	0.1713
P(1,0)	0.1258	0.1083	0.0999	0.0915	0.0846	0.077	0.0714	0.0654
P(1,1)	0.0349	0.0304	0.0282	0.026	0.021	0.0223	0.0205	0.0189
P*(1,1)	0.0349	0.0365	0.0372	0.0379	0.0385	0.0391	0.0397	0.0402
P(2,0)	0.0134	0.0096	0.008	0.0066	0.0056	0.004	0.0039	0.0032
P(2,1)	0.0054	0.004	0.0034	0.0039	0.0025	0.0021	0.0018	0.0015
P*(2,1)	0.0154	0.0151	0.0149	0.0147	0.0146	0.0144	0.0142	0.0141
P(3,0)	0.0013	0.000796	0.000606	0.000453	0.000346	0.000264	0.0002	0.000151
Stable/unstable	Stable	Stable	Stable	Stable	Stable	Stable	Stable	Stable
Average length of queue in first stage	0.0221	0.019896	0.018906	0.019053	0.017446	0.016764	0.0162	0.015751
Average length of queue in system	0.315065	0.305528	0.300541	0.297555	0.287517	0.286902	0.283587	0.279941
Utilization of nurse	0.1	0.12	0.132	0.145	0.16	0.176	0.1936	0.212696
utilization of doctor	0.239925	0.2841	0.309903	0.3373425	0.36836	0.400752	0.43565	0.472664

From Table 3, we see that when there was an increase in service rate by 60% for the first station servers (the nurses), there were a decrease in utilization of the nurse by 37%. The doctors utilization also decreases but it very little >0.01%. As the service rate increase for the nurses, the nurses now served faster and were more idle.

Figure 11 further depicts the results from the table and shows the effect on average length of queue for the first station with variable service rate for the first station servers.

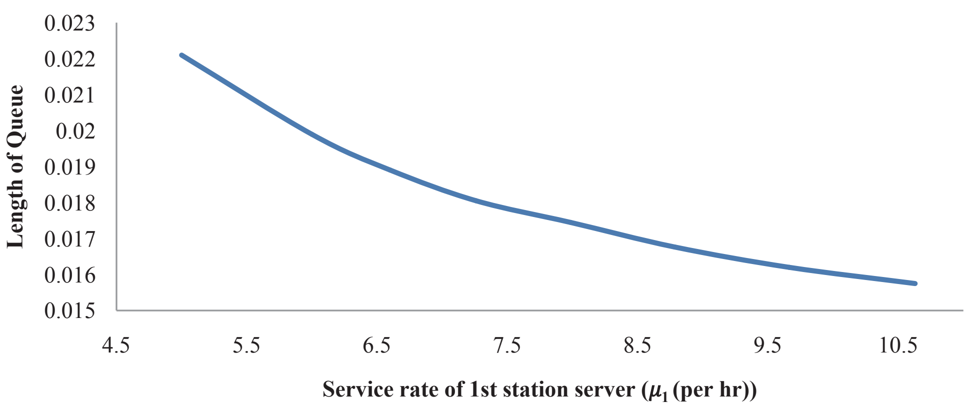

Figure 11
Graph Shows the Effect of Service Rate of the First Stage Server on Length of Queue for Stable System with Fixed K, λ, μ_2 and Variable μ_1

In Figure 11 we observe a similar relation as with the first station server, as the number of patients being served by the second station server increases the length of waiting in queue for the customers decreases.

Similarly we analyzed the effect on average queue length with variable service rate of second stage server; these results are summarized in Table 4.

Table 4
Results Depicting the Effect of Variable μ_2 on the Steady State Probabilities, Average Queue Lengths and Utilization of Servers of System at K = 3, where λ = 1and μ_1 = 5 are Fixed

	μ_2=2	μ_2=3	μ_2=3.2	μ_2=3.3	μ_2=3.6	μ_2=3.8	μ_2=3.96	μ_2=4	μ_2=4.356
P(0,0)	0.4383	0.5519	0.5673	0.5744	0.5939	0.6048	0.6131	0.615	0.631
P(0,1)	0.2191	0.184	0.1773	0.1741	0.1649	0.1592	0.1548	0.1538	0.1449
P(1,0)	0.0925	0.1141	0.1169	0.1183	0.1218	0.1239	0.1254	0.1258	0.1288
P(1,1)	0.0584	0.0442	0.042	0.041	0.0382	0.0365	0.0352	0.0349	0.0325
P*(1,1)	0.0974	0.0552	0.05	0.0476	0.0415	0.038	0.0355	0.0349	0.0303
P(2,0)	0.0102	0.0123	0.0126	0.0127	0.013	0.0132	0.0134	0.0134	0.0137
P(2,1)	0.0098	0.0071	0.0067	0.0065	0.006	0.0056	0.0054	0.0054	0.0049
P*(2,1)	0.0732	0.0302	0.026	0.0242	0.0198	0.0174	0.0158	0.0154	0.0126
P(3,0)	0.001	0.0012	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0014
Stable/unstable	Stable	Stable	Stable	Stable	Stable	Stable	Stable	Stable	Stable
Average length of queue in first stage	0.084	0.0385	0.034	0.032	0.0271	0.0243	0.0225	0.0221	0.0186
Average length of queue in system	0.3925	0.3516	0.3434	0.3395	0.3287	0.3214	0.4642	0.3151	0.305
Utilization of nurse	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
Utilization of doctor	0.4464	0.3126	0.2948	0.2867	0.2645	0.2517	0.2422	0.2399	0.2215

From Table 4, we see that increase in service rate of the second station by 55%, decrease utilization of the doctor by 50% while nurse utilization is steady. When we graphically depict the results in Figure 12, we see that as the service rate of the second stage server is increased the average queue length at the first station decreases.

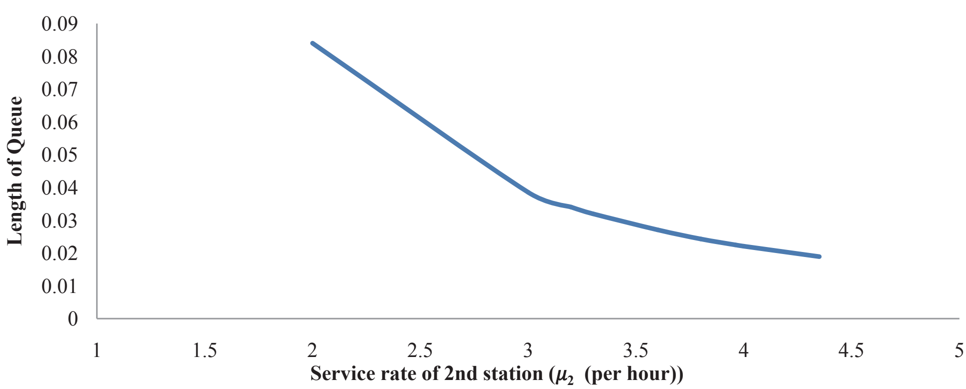

Figure 12
Graph Shows the Effect of Service Rate of the Second Stage Server on Length of Queue for Stable System With Fixed K, λ and μ_1 and Variable μ_2

4. CONCLUSIONS

The nested queue model is a "hierarchical system of multiple queues such that service demand in one queue is created by a subset of the units in the system of another queue higher in the hierarchy of possibly providing an entirely different service" (Modi 1974, p.220).

Our main goal was to utilize a nested queue model for the following cases:

a. To determine the system capacity for a nested model given pre-determined service rates for each station (μ_1 and μ_2) and arrival rates of the patients.

b. To determine the best arrival rate of the patients given a pre-determined system capacity and service rates for each station (μ_1 and μ_2).

c. To determine the best service rate for the server in the first station(μ_1) given a pre-determined system capacity, arrival rate for patients and service rate for the second station (μ_2).

d. To determine the service rate of the server in the second station(μ_2) given a pre-determined system capacity, arrival rate for patients and service rates for the first station (μ_1).

The following conclusions can be drawn.

i. For a pre-determined arrival of the patients and service rates of the servers in both stations, we can determine the number of patients which can be served efficiently. This information can help in scheduling the appointments of the patients.

ii. For a pre-determined number of patients and service rates we can determine the arrival rate so that the system can have efficient performance which again helps in scheduling the appointment of patients.

iii. For a pre-determined number of patients, arrival rate and service rate of one station we can find the service rate for the other station and in order to have to provide the most effective service to the patient.

iv. Further analysis can give us on how a stable system works and how by changing of the different variables in the nested queue system we can achieve the correct utilization of the servers.

The stability analysis of the nested queue model has been applied to a health-care scenario in this paper; however, this analysis can also be applied for other areas e.g. airports, banks etc.

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APPENDIXES

Appendix A

For K= 2

The steady equations are given as follows:

$$\lambda P_{(0,0)} = \mu_2 P_{(0,1)} \tag{1}$$

$$(\lambda+\mu_2)P_{(0,1)} = \mu_2 P*_{(1,1)} + \mu_1 P_{(1,0)} \tag{2}$$

$$(\lambda+\mu_1)P_{(1,0)} = \lambda P_{(0,0)} + \mu_2 P_{(1,1)} \tag{3}$$

$$(\mu_1 + \mu_2)P_{(1,1)} = \lambda P_{(0,1)} + 2\mu_1 P_{(2,0)} \tag{4}$$

$$(\mu_2)P*_{(1,1)} = \mu_1 P_{(1,1)} \tag{5}$$

$$(2\mu_1)P_{(2,0)} = \lambda P_{(1,0)} \tag{6}$$

The sum of the probabilities:

$$P_{(0,0)} + P_{(0,1)} + P_{(1,0)} + P_{(1,1)} + P*_{(1,1)} = 1 \tag{7}$$

Simplifying equations (1), (5) and (6)

$$\lambda P_{(0,0)} = \mu_2 P_{(0,1)}$$

$$P_{(0,1)} = \frac{\lambda}{\mu_2} P_{(0,0)}$$

$$P*_{(1,1)} = \frac{\mu_1}{\mu_2} P_{(1,1)}$$

$$P_{(2,0)} = \frac{\lambda}{2\mu_1} P_{(1,0)}$$

Taking equation (3)

$$(\lambda+\mu_1)P_{(1,0)} = \lambda P_{(0,0)} + \mu_2 P_{(1,1)}$$

$$P_{(1,0)} = \frac{\lambda}{(\lambda+\mu_1)} P_{(0,0)} + \frac{\mu_2}{(\lambda+\mu_1)} P_{(1,1)}$$

Taking equation (2)

$$\mu_1 P_{(1,0)} = (\lambda+\mu_2)P_{(0,1)} - \mu_1 P_{(1,1)}$$

$$\left(\frac{\lambda}{(\lambda+\mu_1)} P_{(0,0)} + \frac{\mu_2}{(\lambda+\mu_1)} P_{(1,1)}\right)\mu_1 = \frac{\lambda(\lambda+\mu_2)}{\mu_2} P_{(0,0)} - \mu_1 P_{(1,1)}$$

$$\frac{\lambda}{(\lambda+\mu_1)} P_{(0,0)} + \frac{\mu_2}{(\lambda+\mu_1)} P_{(1,1)} = \frac{\lambda(\lambda+\mu_2)}{\mu_2\mu_1} P_{(0,0)} - P_{(1,1)}$$

$$\left(\frac{\mu_2}{(\lambda+\mu_1)} + 1\right) P_{(1,1)} = \left(\frac{\lambda(\lambda+\mu_2)}{\mu_2\mu_1} - \frac{\lambda}{(\lambda+\mu_1)}\right) P_{(0,0)}$$

$$\left(\frac{\lambda + \mu_1 + \mu_2}{(\lambda + \mu_1)}\right) P_{(1,1)} = \left(\frac{\lambda(\lambda + \mu_2)(\lambda + \mu_1) - \lambda\mu_2\mu_1}{\mu_2\mu_1(\lambda + \mu_1)}\right) P_{(0,0)}$$

$$P_{(1,1)} = \left(\frac{\lambda(\lambda^2 + \mu_1\lambda + \mu_2\lambda + \mu_1\mu_2) - \lambda\mu_2\mu_1}{\mu_2\mu_1(\lambda + \mu_1 + \mu_2)}\right) P_{(0,0)}$$

$$P_{(1,1)} = \left(\frac{\lambda^2(\lambda + \mu_1 + \mu_2)}{\mu_2\mu_1(\lambda + \mu_1 + \mu_2)}\right) P_{(0,0)}$$

$$P_{(1,1)} = \left(\frac{\lambda^2}{\mu_2\mu_1}\right) P_{(0,0)}$$

Substituting

$$P_{(1,1)} = \frac{\mu_2}{\mu_1} P *_{(1,1)}$$

$$P *_{(1,1)} = \frac{\lambda^2}{{\mu_2}^2} P_{(0,0)}$$

Substituting the above equations into equation (4)

$$(\mu_1 + \mu_2)P_{(1,1)} = \lambda P_{(0,1)} + 2\mu_1 P_{(2,0)}$$

$$2\mu_1 P_{(2,0)} = (\mu_1 + \mu_2)P_{(1,1)} - \lambda P_{(0,1)}$$

$$P_{(2,0)} = \frac{1}{2\mu_1}\left(\left(\frac{\lambda^2(\mu_1 + \mu_2)}{\mu_2\mu_1}\right) - \frac{\lambda^2\mu_1}{\mu_2}\right) P_{(0,0)}$$

$$P_{(2,0)} = \frac{1}{2\mu_1}\left(\frac{\lambda^2}{\mu_1}\right) P_{(0,0)}$$

$$P_{(2,0)} = \frac{\lambda^2}{2{\mu_1}^2} P_{(0,0)}$$

$$P_{(2,0)} = \frac{\lambda}{2\mu_1} P_{(1,0)}$$

$$P_{(1,0)} = \frac{\lambda}{\mu_1} P_{(0,0)}$$

Substituting all values into

$$P_{(0,0)} + P_{(0,1)} + P_{(1,0)} + P_{(1,1)} + P *_{(1,1)} + P_{(2,0)} = 1$$

$$P_{(0,0)} + \frac{\lambda}{\mu_2}P_{(0,0)} + \frac{\lambda}{\mu_1}P_{(0,0)} + \frac{\lambda^2}{\mu_2\mu_1}P_{(0,0)} + \frac{\lambda^2}{{\mu_2}^2}P_{(0,0)} + \frac{\lambda^2}{2{\mu_1}^2}P_{(0,0)} = 1$$

$$\frac{2{\mu_1}^2{\mu_2}^2 + 2\lambda{\mu_1}^2\mu_2 + 2\lambda\mu_1{\mu_2}^2 + 2\lambda^2\mu_1\mu_2 + 2\lambda^2{\mu_1}^2 + \lambda^2{\mu_2}^2}{2{\mu_1}^2{\mu_2}^2}P_{(0,0)} = 1$$

$$P_{(0,0)} = \frac{2{\mu_1}^2{\mu_2}^2}{2{\mu_1}^2{\mu_2}^2 + 2\lambda\mu_1\mu_2(\mu_1+\mu_2) + \lambda^2(2\mu_1\mu_2 + 2{\mu_1}^2 + {\mu_2}^2)}$$

We know that

$$P_{(0,1)} = \frac{\lambda}{\mu_2}P_{(0,0)}$$

$$P_{(0,1)} = \frac{2\lambda{\mu_1}^2\mu_2}{2{\mu_1}^2{\mu_2}^2 + 2\lambda\mu_1\mu_2(\mu_1+\mu_2) + \lambda^2(2\mu_1\mu_2 + 2{\mu_1}^2 + {\mu_2}^2)} \quad (8)$$

$$P_{(1,1)} = \left(\frac{\lambda^2}{\mu_2\mu_1}\right)P_{(0,0)}$$

$$P_{(1,1)} = \frac{2\lambda^2\mu_1\mu_2}{2{\mu_1}^2{\mu_2}^2 + 2\lambda\mu_1\mu_2(\mu_1+\mu_2) + \lambda^2(2\mu_1\mu_2 + 2{\mu_1}^2 + {\mu_2}^2)} \quad (9)$$

$$P*_{(11)} = \frac{\lambda^2}{{\mu_2}^2}P_{(00)}$$

$$P*_{(1,1)} = \frac{2\lambda^2{\mu_1}^2}{2{\mu_1}^2{\mu_2}^2 + 2\lambda\mu_1\mu_2(\mu_1+\mu_2) + \lambda^2(2\mu_1\mu_2 + 2{\mu_1}^2 + {\mu_2}^2)} \quad (10)$$

$$P_{(1,0)} = \frac{\lambda}{\mu_1}P_{(0,0)}$$

$$P_{(1,0)} = \frac{2\lambda\mu_1{\mu_2}^2}{2{\mu_1}^2{\mu_2}^2 + 2\lambda\mu_1\mu_2(\mu_1+\mu_2) + \lambda^2(2\mu_1\mu_2 + 2{\mu_1}^2 + {\mu_2}^2)} \quad (11)$$

$$P_{(2,0)} = \frac{\lambda^2}{2{\mu_1}^2}P_{(00)}$$

$$P_{(2,0)} = \frac{\lambda^2{\mu_2}^2}{2{\mu_1}^2{\mu_2}^2 + 2\lambda\mu_1\mu_2(\mu_1+\mu_2) + \lambda^2(2\mu_1\mu_2 + 2{\mu_1}^2 + {\mu_2}^2)} \quad (12)$$

Appendix B

For K=3

The steady state equations are as follows

$$0 = -\lambda P_{(0,0)} + \mu_2 P_{(0,1)} \quad (1)$$

$$0 = -(\lambda+\mu_2)P_{(0,1)} + \mu_2 P*_{(1,1)} + \mu_1 P_{(1,0)} \quad (2)$$

$$0 = -(\lambda+\mu_1)P_{(10)} + \lambda P_{(00)} + \mu_2 P_{(11)} \quad (3)$$

$$0 = -(\lambda+\mu_2)P*_{(11)} + \mu_1 P_{(11)} \quad (4)$$

$$0 = -(\lambda+\mu_1 + \mu_2)P_{(1,1)} + 2\mu_1 P_{(2,0)} + \lambda P_{(0,1)} + \mu_2 P*_{(2,1)} \quad (5)$$

$$0 = -(\lambda+2\mu_1)P_{(2,0)} + \lambda P_{(1,0)} + \mu_2 P_{(2,1)} \quad (6)$$

$$0 = -(\mu_2)P*_{(2,1)} + \mu_1 P_{(2,1)} + \lambda P*_{(1,1)} \quad (7)$$

$$0 = -(\mu_1 + \mu_2)P_{(2,1)} + 2\mu_1 P_{(3,0)} + \lambda P_{(1,1)} \quad (8)$$

$$0 = -(2\mu_1)P_{(3,0)} + \lambda P_{(2,0)} \quad (9)$$

Solving

$$P_{(0,1)} = \frac{\lambda}{\mu_2} P_{(0,0)} \quad (10)$$

$$P*_{(1,1)} = \frac{\mu_1}{\mu_2} P_{(1,1)} \quad (11)$$

$$P_{(3,0)} = \frac{\lambda}{2\mu_1} P_{(2,0)} \quad (12)$$

Taking equation (15)

$$(\lambda+\mu_1)P_{(1,0)} = \lambda P_{(0,0)} + \mu_2 P_{(1,1)}$$

$$P_{(1,0)} = \frac{\lambda}{(\lambda+\mu_1)} P_{(0,0)} + \frac{\mu_2}{(\lambda+\mu_1)} P_{(1,1)}$$

Taking equation (14)

$$\mu_1 P_{(1,0)} = (\lambda+\mu_2)P_{(0,1)} - \mu_1 P_{(1,1)}$$

$$\left(\frac{\lambda}{(\lambda+\mu_1)} P_{(0,0)} + \frac{\mu_2}{(\lambda+\mu_1)} P_{(1,1)}\right)\mu_1 = \frac{\lambda(\lambda+\mu_2)}{\mu_2} P_{(0,0)} - \mu_1 P_{(1,1)}$$

$$\frac{\lambda}{(\lambda+\mu_1)} P_{(0,0)} + \frac{\mu_2}{(\lambda+\mu_1)} P_{(1,1)} = \frac{\lambda(\lambda+\mu_2)}{\mu_2\mu_1} P_{(0,0)} - P_{(1,1)}$$

$$\left(\frac{\mu_2}{(\lambda+\mu_1)} + 1\right) P_{(1,1)} = \left(\frac{\lambda(\lambda+\mu_2)}{\mu_2\mu_1} - \frac{\lambda}{(\lambda+\mu_1)}\right) P_{(0,0)}$$

$$\left(\frac{\lambda+\mu_1+\mu_2}{(\lambda+\mu_1)}\right) P_{(1,1)} = \left(\frac{\lambda(\lambda+\mu_2)(\lambda+\mu_1) - \lambda\mu_2\mu_1}{\mu_2\mu_1(\lambda+\mu_1)}\right) P_{(0,0)}$$

$$P_{(1,1)} = \left(\frac{\lambda(\lambda^2+\mu_1\lambda+\mu_2\lambda+\mu_1\mu_2) - \lambda\mu_2\mu_1}{\mu_2\mu_1(\lambda+\mu_1+\mu_2)}\right) P_{(0,0)}$$

$$P_{(1,1)} = \left(\frac{\lambda^2(\lambda+\mu_1+\mu_2)}{\mu_2\mu_1(\lambda+\mu_1+\mu_2)}\right) P_{(0,0)}$$

$$P_{(1,1)} = \left(\frac{\lambda^2}{\mu_2\mu_1}\right) P_{(0,0)} \quad (13)$$

Substituting

$$P_{(1,1)} = \frac{\mu_2}{\mu_1} P*_{(1,1)}$$

$$P*_{(1,1)} = \frac{\lambda^2}{{\mu_2}^2} P_{(0,0)} \tag{14}$$

Substituting $P_{(11)}$ into equation (15) we can find $P_{(10)}$

$$0 = -(\lambda+\mu_1)P_{(1,0)} + \lambda P_{(0,0)} + \mu_2 P_{(1,1)}$$

$$(\lambda+\mu_1)P_{(1,0)} = \lambda P_{(0,0)} + \left(\frac{\lambda^2}{\mu_1}\right) P_{(0,0)}$$

$$P_{(1,0)} = \left(\frac{\lambda\mu_1+\lambda^2}{\mu_1(\lambda+\mu_1)}\right) P_{(0,0)} \tag{15}$$

Now to find $P_{(2,0)}$, $P_{(2,1)}$ and $P*_{(2,1)}$. Substitute into equation (19) the values for $P*_{(1,1)}$

$$0 = -(\mu_2)P*_{(2,1)} + \mu_1 P_{(2,1)} + \frac{\lambda^3}{{\mu_2}^2} P_{(0,0)}$$

$$P*_{(2,1)} = \frac{\mu_1}{\mu_2} P_{(2,1)} + \frac{\lambda^3}{{\mu_2}^3} P_{(0,0)} \tag{16}$$

Substitute into equation (18) the values for $P_{(1,0)}$

$$0 = -(\lambda+2\mu_1)P_{(2,0)} + \left(\frac{\lambda^2\mu_1 + \lambda^2}{\mu_1(\lambda+\mu_1)}\right) P_{(00)} + \mu_2 P_{(2,1)}$$

$$P_{(2,0)} = \left(\frac{\lambda^2\mu_1+\lambda^2}{\mu_1(\lambda+2\mu_1)(\lambda+\mu_1)}\right) P_{(00)} + \frac{\mu_2}{(\lambda+2\mu_1)} P_{(2,1)} \tag{17}$$

Substituting equations (29) and (30) into (17)

$$0 = -(\lambda+\mu_1+\mu_2)\left(\frac{\lambda^2}{\mu_2\mu_1}\right) P_{(00)} + 2\left(\frac{\lambda^2\mu_1+\lambda^2}{\mu_1(\lambda+2\mu_1)(\lambda+\mu_1)}\right) P_{(00)} + \frac{2\mu_1\mu_2}{(\lambda+2\mu_1)} P_{(2,1)} + \frac{\lambda^2}{\mu_2} P_{(00)} + \mu_1 P_{(2,1)} + \frac{\lambda^3}{{\mu_2}^2} P_{(00)}$$

Simplifying

$$P_{(2,1)} = \left(\frac{(\lambda+2\mu_1)}{2\mu_1\mu_2+\mu_1(\lambda+2\mu_1)}\left(\frac{\lambda^2(\lambda+\mu_1+\mu_2)}{(\mu_2\mu_1)} - 2\left(\frac{\lambda^2\mu_1+\lambda^2}{\mu_1(\lambda+2\mu_1)(\lambda+\mu_1)}\right) - \frac{\lambda^2}{\mu_2} - \frac{\lambda^3}{{\mu_2}^2}\right)\right) P_{(0,0)} \tag{18}$$

$$P_{(2,0)} = \left(\left(\frac{\lambda^2\mu_1+\lambda^2}{\mu_1(\lambda+2\mu_1)(\lambda+\mu_1)}\right) + \frac{\mu_2}{(\lambda+2\mu_1)}\left(\frac{(\lambda+2\mu_1)}{2\mu_1\mu_2+\mu_1(\lambda+2\mu_1)}\left(\frac{\lambda^2(\lambda+\mu_1+\mu_2)}{(\mu_2\mu_1)} - 2\left(\frac{\lambda^2\mu_1+\lambda^2}{\mu_1(\lambda+2\mu_1)(\lambda+\mu_1)}\right) - \frac{\lambda^2}{\mu_2} - \frac{\lambda^3}{{\mu_2}^2}\right)\right)\right) P_{(0,0)} \tag{19}$$

$$P*_{(2,1)} = \left(\frac{\mu_1}{\mu_2}\left(\frac{(\lambda+2\mu_1)}{2\mu_1\mu_2+\mu_1(\lambda+2\mu_1)}\left(\frac{\lambda^2(\lambda+\mu_1+\mu_2)}{(\mu_2\mu_1)} - 2\left(\frac{\lambda^2\mu_1+\lambda^2}{\mu_1(\lambda+2\mu_1)(\lambda+\mu_1)}\right) - \frac{\lambda^2}{\mu_2} - \frac{\lambda^3}{{\mu_2}^2}\right)\right) + \frac{\lambda^3}{{\mu_2}^3}\right) P_{(0,0)} \tag{20}$$

Substituting equation (32) into (25)

$$P_{(3,0)} = \frac{\lambda}{2\mu_1}\left(\left(\frac{\lambda^2\mu_1+\lambda^2}{\mu_1(\lambda+2\mu_1)(\lambda+\mu_1)}\right) + \frac{\mu_2}{(\lambda+2\mu_1)}\left(\frac{(\lambda+2\mu_1)}{2\mu_1\mu_2+\mu_1(\lambda+2\mu_1)}\left(\frac{\lambda^2(\lambda+\mu_1+\mu_2)}{(\mu_2\mu_1)} - 2\left(\frac{\lambda^2\mu_1+\lambda^2}{\mu_1(\lambda+2\mu_1)(\lambda+\mu_1)}\right) - \frac{\lambda^2}{\mu_2} - \frac{\lambda^3}{{\mu_2}^2}\right)\right)\right) P_{(0,0)} \tag{21}$$

Solving for $P_{(00)}$

$$P_{(0,0)} + P_{(0,1)} + P_{(1,0)} + P_{(1,1)} + P*_{(1,1)} + P_{(2,0)} + P_{(2,1)} + P*_{(2,1)} + P_{(3,0)} = 1$$

$$P_{(0,0)} = \frac{4\lambda^2\mu_1{}^3\mu_2{}^3 + 10\lambda\mu_1{}^4\mu_2{}^3 + 10\lambda\mu_1{}^3\mu_2{}^4 + 4\mu_1{}^5\mu_2{}^3 + 8\mu_1{}^4\mu_2{}^4 + 4\mu_1{}^3\mu_2{}^5}{\begin{array}{c} 2\lambda^6\mu_1{}^2 + 2\lambda^6\mu_1\mu_2 + \lambda^6\mu_2{}^2 + 8\lambda^5\mu_1{}^3 + 14\lambda^5\mu_1{}^2\mu_2 + 10\lambda^5\mu_1\mu_2{}^2 + 3\lambda^5\mu_2{}^3 + 10\lambda^4\mu_1{}^4 + 28\lambda^4\mu_1{}^3\mu_2 \\ +30\lambda^4\mu_1{}^2\mu_2{}^2 + 15\lambda^4\mu_1\mu_2{}^3 + 3\lambda^4\mu_2{}^4 + 4\lambda^3\mu_1{}^5 + 22\lambda^3\mu_1{}^4\mu_2 + 42\lambda^3\mu_1{}^3\mu_2{}^2 + 33\lambda^3\mu_1{}^2\mu_2{}^3 + 10\lambda^3\mu_1\mu_2{}^4 \\ +\lambda^3\mu_2{}^5 + 4\lambda^2\mu_1{}^5\mu_2 + 22\lambda^2\mu_1{}^4\mu_2{}^2 + 38\lambda^2\mu_1{}^3\mu_2{}^3 + 18\lambda^2\mu_1{}^2\mu_2{}^4 + 2\lambda^2\mu_1\mu_2{}^5 + 4\lambda\mu_1{}^5\mu_2{}^2 \\ 22\lambda\mu_1{}^4\mu_2{}^3 + 22\lambda\mu_1{}^3\mu_2{}^4 + 4\lambda\mu_1{}^2\mu_2{}^5 + 4\mu_1{}^5\mu_2{}^3 + 8\mu_1{}^4\mu_2{}^4 + 4\mu_1{}^3\mu_2{}^5 \end{array}} \quad (22)$$

Substituting $P_{(0,0)}$ into the above equations we find the additional probability values.

$$P_{(0,1)} = \frac{\lambda(4\lambda^2\mu_1{}^3\mu_2{}^3 + 10\lambda\mu_1{}^4\mu_2{}^3 + 10\lambda\mu_1{}^3\mu_2{}^4 + 4\mu_1{}^5\mu_2{}^3 + 8\mu_1{}^4\mu_2{}^4 + 4\mu_1{}^3\mu_2{}^5)}{\mu_2\left(\begin{array}{c} 2\lambda^6\mu_1{}^2 + 2\lambda^6\mu_1\mu_2 + \lambda^6\mu_2{}^2 + 8\lambda^5\mu_1{}^3 + 14\lambda^5\mu_1{}^2\mu_2 + 10\lambda^5\mu_1\mu_2{}^2 + 3\lambda^5\mu_2{}^3 + 10\lambda^4\mu_1{}^4 \\ +28\lambda^4\mu_1{}^3\mu_2 \\ +30\lambda^4\mu_1{}^2\mu_2{}^2 + 15\lambda^4\mu_1\mu_2{}^3 + 3\lambda^4\mu_2{}^4 + 4\lambda^3\mu_1{}^5 + 22\lambda^3\mu_1{}^4\mu_2 + 42\lambda^3\mu_1{}^3\mu_2{}^2 + 33\lambda^3\mu_1{}^2\mu_2{}^3 + \\ 10\lambda^3\mu_1\mu_2{}^4 \\ +\lambda^3\mu_2{}^5 + 4\lambda^2\mu_1{}^5\mu_2 + 22\lambda^2\mu_1{}^4\mu_2{}^2 + 38\lambda^2\mu_1{}^3\mu_2{}^3 + 18\lambda^2\mu_1{}^2\mu_2{}^4 + 2\lambda^2\mu_1\mu_2{}^5 + 4\lambda\mu_1{}^5\mu_2{}^2 \\ 22\lambda\mu_1{}^4\mu_2{}^3 + 22\lambda\mu_1{}^3\mu_2{}^4 + 4\lambda\mu_1{}^2\mu_2{}^5 + 4\mu_1{}^5\mu_2{}^3 + 8\mu_1{}^4\mu_2{}^4 + 4\mu_1{}^3\mu_2{}^5 \end{array}\right)} \quad (23)$$

$$P_{(1,0)} = \frac{2\lambda\mu_1{}^2\mu_2{}^2(\lambda^3 + 2\lambda^2\mu_1 + 4\lambda^2\mu_2 + 5\lambda\mu_1\mu_2 + 5\lambda\mu_2{}^2 + 2\mu_1{}^2\mu_2 + 4\mu_1\mu_2{}^2 + 2\mu_2{}^3}{\begin{array}{c} 2\lambda^6\mu_1{}^2 + 2\lambda^6\mu_1\mu_2 + \lambda^6\mu_2{}^2 + 8\lambda^5\mu_1{}^3 + 14\lambda^5\mu_1{}^2\mu_2 + 10\lambda^5\mu_1\mu_2{}^2 + 3\lambda^5\mu_2{}^3 + 10\lambda^4\mu_1{}^4 + 28\lambda^4\mu_1{}^3\mu_2 \\ +30\lambda^4\mu_1{}^2\mu_2{}^2 + 15\lambda^4\mu_1\mu_2{}^3 + 3\lambda^4\mu_2{}^4 + 4\lambda^3\mu_1{}^5 + 22\lambda^3\mu_1{}^4\mu_2 + 42\lambda^3\mu_1{}^3\mu_2{}^2 + 33\lambda^3\mu_1{}^2\mu_2{}^3 + 10\lambda^3\mu_1\mu_2{}^4 \\ +\lambda^3\mu_2{}^5 + 4\lambda^2\mu_1{}^5\mu_2 + 22\lambda^2\mu_1{}^4\mu_2{}^2 + 38\lambda^2\mu_1{}^3\mu_2{}^3 + 18\lambda^2\mu_1{}^2\mu_2{}^4 + 2\lambda^2\mu_1\mu_2{}^5 + 4\lambda\mu_1{}^5\mu_2{}^2 \\ 22\lambda\mu_1{}^4\mu_2{}^3 + 22\lambda\mu_1{}^3\mu_2{}^4 + 4\lambda\mu_1{}^2\mu_2{}^5 + 4\mu_1{}^5\mu_2{}^3 + 8\mu_1{}^4\mu_2{}^4 + 4\mu_1{}^3\mu_2{}^5 \end{array}} \quad (24)$$

$$P_{(1,1)} = \frac{(2\lambda^2\mu_1{}^2\mu_2(\lambda + \mu_2)(\lambda^2 + 3\lambda\mu_1 + 3\lambda\mu_2 + 2\mu_1{}^2 + 4\mu_1\mu_2 + 2\mu_2{}^2))}{\begin{array}{c} 2\lambda^6\mu_1{}^2 + 2\lambda^6\mu_1\mu_2 + \lambda^6\mu_2{}^2 + 8\lambda^5\mu_1{}^3 + 14\lambda^5\mu_1{}^2\mu_2 + 10\lambda^5\mu_1\mu_2{}^2 + 3\lambda^5\mu_2{}^3 + 10\lambda^4\mu_1{}^4 + 28\lambda^4\mu_1{}^3\mu_2 \\ +30\lambda^4\mu_1{}^2\mu_2{}^2 + 15\lambda^4\mu_1\mu_2{}^3 + 3\lambda^4\mu_2{}^4 + 4\lambda^3\mu_1{}^5 + 22\lambda^3\mu_1{}^4\mu_2 + 42\lambda^3\mu_1{}^3\mu_2{}^2 + 33\lambda^3\mu_1{}^2\mu_2{}^3 + 10\lambda^3\mu_1\mu_2{}^4 \\ +\lambda^3\mu_2{}^5 + 4\lambda^2\mu_1{}^5\mu_2 + 22\lambda^2\mu_1{}^4\mu_2{}^2 + 38\lambda^2\mu_1{}^3\mu_2{}^3 + 18\lambda^2\mu_1{}^2\mu_2{}^4 + 2\lambda^2\mu_1\mu_2{}^5 + 4\lambda\mu_1{}^5\mu_2{}^2 \\ 22\lambda\mu_1{}^4\mu_2{}^3 + 22\lambda\mu_1{}^3\mu_2{}^4 + 4\lambda\mu_1{}^2\mu_2{}^5 + 4\mu_1{}^5\mu_2{}^3 + 8\mu_1{}^4\mu_2{}^4 + 4\mu_1{}^3\mu_2{}^5 \end{array}} \quad (25)$$

$$P\,*_{(1,1)} = \frac{(2\lambda^2\mu_1{}^3\mu_2(\lambda^2 + 3\lambda\mu_1 + 3\lambda\mu_2 + 2\mu_1{}^2 + 4\mu_1\mu_2 + 2\mu_2{}^2))}{\begin{array}{c} 2\lambda^6\mu_1{}^2 + 2\lambda^6\mu_1\mu_2 + \lambda^6\mu_2{}^2 + 8\lambda^5\mu_1{}^3 + 14\lambda^5\mu_1{}^2\mu_2 + 10\lambda^5\mu_1\mu_2{}^2 + 3\lambda^5\mu_2{}^3 + 10\lambda^4\mu_1{}^4 + 28\lambda^4\mu_1{}^3\mu_2 \\ +30\lambda^4\mu_1{}^2\mu_2{}^2 + 15\lambda^4\mu_1\mu_2{}^3 + 3\lambda^4\mu_2{}^4 + 4\lambda^3\mu_1{}^5 + 22\lambda^3\mu_1{}^4\mu_2 + 42\lambda^3\mu_1{}^3\mu_2{}^2 + 33\lambda^3\mu_1{}^2\mu_2{}^3 + 10\lambda^3\mu_1\mu_2{}^4 \\ +\lambda^3\mu_2{}^5 + 4\lambda^2\mu_1{}^5\mu_2 + 22\lambda^2\mu_1{}^4\mu_2{}^2 + 38\lambda^2\mu_1{}^3\mu_2{}^3 + 18\lambda^2\mu_1{}^2\mu_2{}^4 + 2\lambda^2\mu_1\mu_2{}^5 + 4\lambda\mu_1{}^5\mu_2{}^2 \\ 22\lambda\mu_1{}^4\mu_2{}^3 + 22\lambda\mu_1{}^3\mu_2{}^4 + 4\lambda\mu_1{}^2\mu_2{}^5 + 4\mu_1{}^5\mu_2{}^3 + 8\mu_1{}^4\mu_2{}^4 + 4\mu_1{}^3\mu_2{}^5 \end{array}} \quad (26)$$

$$P_{(2,0)} = \frac{(2\lambda^2\mu_1\mu_2{}^2(\lambda^3 + 2\lambda^2\mu_1 + 3\lambda^2\mu_2 + 3\lambda\mu_1\mu_2 + 3\lambda\mu_1{}^2 + \mu_1{}^2\mu_2 + 2\mu_1\mu_2{}^2 + \mu_2{}^3))}{\begin{array}{c} 2\lambda^6\mu_1{}^2 + 2\lambda^6\mu_1\mu_2 + \lambda^6\mu_2{}^2 + 8\lambda^5\mu_1{}^3 + 14\lambda^5\mu_1{}^2\mu_2 + 10\lambda^5\mu_1\mu_2{}^2 + 3\lambda^5\mu_2{}^3 + 10\lambda^4\mu_1{}^4 + 28\lambda^4\mu_1{}^3\mu_2 \\ +30\lambda^4\mu_1{}^2\mu_2{}^2 + 15\lambda^4\mu_1\mu_2{}^3 + 3\lambda^4\mu_2{}^4 + 4\lambda^3\mu_1{}^5 + 22\lambda^3\mu_1{}^4\mu_2 + 42\lambda^3\mu_1{}^3\mu_2{}^2 + 33\lambda^3\mu_1{}^2\mu_2{}^3 + 10\lambda^3\mu_1\mu_2{}^4 \\ +\lambda^3\mu_2{}^5 + 4\lambda^2\mu_1{}^5\mu_2 + 22\lambda^2\mu_1{}^4\mu_2{}^2 + 38\lambda^2\mu_1{}^3\mu_2{}^3 + 18\lambda^2\mu_1{}^2\mu_2{}^4 + 2\lambda^2\mu_1\mu_2{}^5 + 4\lambda\mu_1{}^5\mu_2{}^2 \\ 22\lambda\mu_1{}^4\mu_2{}^3 + 22\lambda\mu_1{}^3\mu_2{}^4 + 4\lambda\mu_1{}^2\mu_2{}^5 + 4\mu_1{}^5\mu_2{}^3 + 8\mu_1{}^4\mu_2{}^4 + 4\mu_1{}^3\mu_2{}^5 \end{array}} \quad (27)$$

$$P_{(2,1)} = \frac{(2\lambda^3\mu_1\mu_2(\lambda^3 + 3\lambda^2\mu_1 + 3\lambda\mu_2{}^2 + \mu_2{}^3) + 2\lambda^3\mu_1{}^2\mu_2(3\lambda\mu_1{}^2 + 5\lambda\mu_2 + 3\mu_2{}^2) + 2\lambda^3\mu_1{}^3\mu_2(2\lambda + 2\mu_2))}{\begin{array}{c} 2\lambda^6\mu_1{}^2 + 2\lambda^6\mu_1\mu_2 + \lambda^6\mu_2{}^2 + 8\lambda^5\mu_1{}^3 + 14\lambda^5\mu_1{}^2\mu_2 + 10\lambda^5\mu_1\mu_2{}^2 + 3\lambda^5\mu_2{}^3 + 10\lambda^4\mu_1{}^4 + 28\lambda^4\mu_1{}^3\mu_2 \\ +30\lambda^4\mu_1{}^2\mu_2{}^2 + 15\lambda^4\mu_1\mu_2{}^3 + 3\lambda^4\mu_2{}^4 + 4\lambda^3\mu_1{}^5 + 22\lambda^3\mu_1{}^4\mu_2 + 42\lambda^3\mu_1{}^3\mu_2{}^2 + 33\lambda^3\mu_1{}^2\mu_2{}^3 + 10\lambda^3\mu_1\mu_2{}^4 \\ +\lambda^3\mu_2{}^5 + 4\lambda^2\mu_1{}^5\mu_2 + 22\lambda^2\mu_1{}^4\mu_2{}^2 + 38\lambda^2\mu_1{}^3\mu_2{}^3 + 18\lambda^2\mu_1{}^2\mu_2{}^4 + 2\lambda^2\mu_1\mu_2{}^5 + 4\lambda\mu_1{}^5\mu_2{}^2 \\ 22\lambda\mu_1{}^4\mu_2{}^3 + 22\lambda\mu_1{}^3\mu_2{}^4 + 4\lambda\mu_1{}^2\mu_2{}^5 + 4\mu_1{}^5\mu_2{}^3 + 8\mu_1{}^4\mu_2{}^4 + 4\mu_1{}^3\mu_2{}^5 \end{array}} \quad (28)$$

$$P\,*_{(2,1)} = \frac{(2\lambda^3\mu_1{}^2(\lambda^3 + 4\lambda^2\mu_1 + 3\lambda^2\mu_2 + 5\lambda\mu_1{}^2 + 8\lambda\mu_1\mu_2 + 3\lambda\,\mu_2{}^2 + 2\mu_1{}^3 + 6\mu_1{}^2\mu_2 + 5\mu_1\mu_2{}^2 + \mu_2{}^3))}{\begin{array}{c} 2\lambda^6\mu_1{}^2 + 2\lambda^6\mu_1\mu_2 + \lambda^6\mu_2{}^2 + 8\lambda^5\mu_1{}^3 + 14\lambda^5\mu_1{}^2\mu_2 + 10\lambda^5\mu_1\mu_2{}^2 + 3\lambda^5\mu_2{}^3 + 10\lambda^4\mu_1{}^4 + 28\lambda^4\mu_1{}^3\mu_2 \\ +30\lambda^4\mu_1{}^2\mu_2{}^2 + 15\lambda^4\mu_1\mu_2{}^3 + 3\lambda^4\mu_2{}^4 + 4\lambda^3\mu_1{}^5 + 22\lambda^3\mu_1{}^4\mu_2 + 42\lambda^3\mu_1{}^3\mu_2{}^2 + 33\lambda^3\mu_1{}^2\mu_2{}^3 + 10\lambda^3\mu_1\mu_2{}^4 \\ +\lambda^3\mu_2{}^5 + 4\lambda^2\mu_1{}^5\mu_2 + 22\lambda^2\mu_1{}^4\mu_2{}^2 + 38\lambda^2\mu_1{}^3\mu_2{}^3 + 18\lambda^2\mu_1{}^2\mu_2{}^4 + 2\lambda^2\mu_1\mu_2{}^5 + 4\lambda\mu_1{}^5\mu_2{}^2 \\ 22\lambda\mu_1{}^4\mu_2{}^3 + 22\lambda\mu_1{}^3\mu_2{}^4 + 4\lambda\mu_1{}^2\mu_2{}^5 + 4\mu_1{}^5\mu_2{}^3 + 8\mu_1{}^4\mu_2{}^4 + 4\mu_1{}^3\mu_2{}^5 \end{array}} \quad (29)$$

$$P_{(3,0)} = \frac{\lambda}{2\mu_1}\left(\left(\frac{\lambda^2\mu_1+\lambda^2}{\mu_1(\lambda+2\mu_1)(\lambda+\mu_1)}\right) + \frac{\mu_2}{(\lambda+2\mu_1)}\left(\frac{(\lambda+2\mu_1)}{2\mu_1\mu_2+\mu_1(\lambda+2\mu_1)}\left(\frac{\lambda^2(\lambda+\mu_1+\mu_2)}{(\mu_2\mu_1)} - 2\left(\frac{\lambda^2\mu_1+\lambda^2}{\mu_1(\lambda+2\mu_1)(\lambda+\mu_1)}\right) - \frac{\lambda^2}{\mu_2} - \frac{\lambda^3}{\mu_2{}^2}\right)\right)\right)$$

$$\left(\frac{4\lambda^2\mu_1{}^3\mu_2{}^3 + 10\lambda\mu_1{}^4\mu_2{}^3 + 10\lambda\mu_1{}^3\mu_2{}^4 + 4\mu_1{}^5\mu_2{}^3 + 8\mu_1{}^4\mu_2{}^4 + 4\mu_1{}^3\mu_2{}^5}{\begin{array}{c} 2\lambda^6\mu_1{}^2 + 2\lambda^6\mu_1\mu_2 + \lambda^6\mu_2{}^2 + 8\lambda^5\mu_1{}^3 + 14\lambda^5\mu_1{}^2\mu_2 + 10\lambda^5\mu_1\mu_2{}^2 + 3\lambda^5\mu_2{}^3 + 10\lambda^4\mu_1{}^4 + 28\lambda^4\mu_1{}^3\mu_2 \\ +30\lambda^4\mu_1{}^2\mu_2{}^2 + 15\lambda^4\mu_1\mu_2{}^3 + 3\lambda^4\mu_2{}^4 + 4\lambda^3\mu_1{}^5 + 22\lambda^3\mu_1{}^4\mu_2 + 42\lambda^3\mu_1{}^3\mu_2{}^2 + 33\lambda^3\mu_1{}^2\mu_2{}^3 + 10\lambda^3\mu_1\mu_2{}^4 \\ +\lambda^3\mu_2{}^5 + 4\lambda^2\mu_1{}^5\mu_2 + 22\lambda^2\mu_1{}^4\mu_2{}^2 + 38\lambda^2\mu_1{}^3\mu_2{}^3 + 18\lambda^2\mu_1{}^2\mu_2{}^4 + 2\lambda^2\mu_1\mu_2{}^5 + 4\lambda\mu_1{}^5\mu_2{}^2 \\ 22\lambda\mu_1{}^4\mu_2{}^3 + 22\lambda\mu_1{}^3\mu_2{}^4 + 4\lambda\mu_1{}^2\mu_2{}^5 + 4\mu_1{}^5\mu_2{}^3 + 8\mu_1{}^4\mu_2{}^4 + 4\mu_1{}^3\mu_2{}^5 \end{array}}\right) \quad (30)$$

Appendix C

Table 1-AC
Results Depicting Steady State Probabilities for Average Queue Lengths and Stability Analysis for Values of K = 14 and 15

	K=14, λ=1, μ_1=3 and μ_2=2	K=15, λ=1, μ_1=3 and μ_2=2	K=15, λ=0.4, μ_1=1 and μ_2=2	K=15, λ=0.5, μ_1=1 and μ_2=2	K=15, λ=0.6, μ_1=1 and μ_2=2
P(0,0)	0.338955	0.338938	0.624025	0.5245721	0.3538936
P(0,1)	0.169477	0.169469	0.0936037	0.1049144	0.1061681
P(1,0)	0.121055	0.121049	0.1895476	0.2142463	0.2214363
P(1,1)	0.072633	0.07263	0.0296022	0.045058	0.0709809
P*(1,1)	0.072633	0.07263	0.0128705	0.0187742	0.0273004
P(2,0)	0.026073	0.026072	0.0303241	0.0468937	0.0273004
P(2,1)	0.030729	0.030728	0.0064405	0.0134232	0.0333792
P*(2,1)	0.05494	0.054938	0.004479	0.008722	0.0191383
P(3,0)	0.007784	0.007783	0.0050533	0.0108413	0.0288205
P(3,1)	0.014207	0.014206	0.0012627	0.0036308	0.0144335
P*(3,1)	0.03252	0.032519	0.0011332	0.0029665	0.0099679
P(4,0)	0.003124	0.003124	0.0008661	0.0025978	0.0113278
P(4,1)	0.007043	0.007042	0.000238	0.0009491	0.00608
P*(4,1)	0.017883	0.017882	0.0002513	0.0008899	0.0046387
P(5,0)	0.001481	0.001481	0.0001513	0.000637	0.004567
P(5,1)	0.003621	0.003621	0.0000441	0.0002449	0.0025388
P*(5,1)	0.009582	0.009582	0.000052	0.0002503	0.0020469
P(6,0)	0.000752	0.000752	0.0000268	0.0001585	0.001867
P(6,1)	0.001892	0.001892	0.0000081	0.0000628	0.0010569
P*(6,1)	0.005086	0.005086	0.0000103	0.0000679	0.0008789
P(7,0)	0.000392	0.000392	0.0000048	0.0000398	0.000769
P(7,1)	0.000995	0.000995	0.0000015	0.0000161	0.0004396
P*(7,1)	0.00269	0.00269	0.000002	0.000018	0.0003719
P(8,0)	0.000206	0.000206	0.0000009	0.0000101	0.000318
P(8,1)	0.000524	0.000524	0.0000003	0.0000041	0.0001827
P*(8,1)	0.001421	0.001421	0.0000004	0.0000047	0.0001561
P(9,0)	0.000108	0.000108	0.0000002	0.0000026	0.0001318
P(9,1)	0.000277	0.000277	0.0000001	0.0000011	0.000076
P*(9,1)	0.00075	0.00075	0.0000001	0.0000012	0.0000652
P(10,0)	0.000057	0.000057	0	0.0000007	0.0000547
P(10,1)	0.000146	0.000146	0	0.0000003	0.0000316
P*(10,1)	0.000396	0.000396	0	0.0000003	0.0000272
P(11,0)	0.00003	0.00003	0	0.0000002	0.0000227
P(11,1)	0.000077	0.000077	0	0.0000001	0.0000131
P*(11,1)	0.000209	0.000209	0	0.0000001	0.0000113
P(12,0)	0.000016	0.000016	0	0	0.0000094
P(12,1)	0.000041	0.000041	0	0	0.0000055
P*(12,1)	0.00011	0.00011	0	0	0.0000047
P(13,0)	0.000005	0.000008	0	0	0.0000039
P(13,1)	0.000009	0.000021	0	0	0.0000023
P*(13,1)	0.000069	0.000058	0	0	0.000002
P(14,0)	0.000001	0.000003	0	0	0.0000015
P(14,1)		0.000005	0	0	0.0000007
P*(14,1)		0.000036	0	0	0.000001
P(15,0)		0	0	0	0.0000004
Stable/Unstable	Stable	Unstable	Unstable	Unstable	Stable
Average length of queue in first stage	0.422267	0.422855	0.0250605	0.0626502	0.2509502
Average length of queue in System	1.453823	1.454461	0.4938594	0.6943455	1.1166054

Table 2-AC
Results Depicting Steady State Probabilities for Average Queue Lengths and Stability Analysis for Values of K = 12

	K=12, λ=1, μ_1=11.4 and μ_2=2	K=12, λ=1, μ_1=11.5 and μ_2=2	K=12, λ=1, μ_1=3 and μ_2=4.4	K=12, λ=1, μ_1=3 and μ_2=4.5
P(0,0)	0.4558702	0.4562533	0.5440102	0.5478179
P(0,1)	0.2279351	0.2281267	0.1236387	0.1217373
P(1,0)	0.0412869	0.040954	0.1857211	0.1868775
P(1,1)	0.028044	0.0278359	0.0451987	0.044376
P*(1,1)	0.1065671	0.1067044	0.0251104	0.0242051
P(2,0)	0.0023114	0.0022726	0.0342554	0.0343839
P(2,1)	0.0068623	0.0067941	0.0122879	0.0119578
P*(2,1)	0.0615989	0.0616123	0.0114767	0.0109233
P(3,0)	0.0003399	0.0003332	0.006846	0.0068325
P(3,1)	0.0028891	0.0028619	0.0031061	0.0029874
P*(3,1)	0.0315115	0.031508	0.0038509	0.0036156
P(4,0)	0.0001327	0.0001302	0.0014631	0.0014467
P(4,1)	0.0014089	0.001396	0.0007718	0.0007321
P*(4,1)	0.0158577	0.015854	0.0011419	0.0010567
P(5,0)	0.0000647	0.0000636	0.0003296	0.0003219
P(5,1)	0.0007041	0.0006977	0.0001918	0.0001792
P*(5,1)	0.0079616	0.0079591	0.000318	0.0002899
P(6,0)	0.0000324	0.0000318	0.0000772	0.0000743
P(6,1)	0.0003532	0.0003499	0.0000479	0.000044
P*(6,1)	0.0039959	0.0039944	0.0000855	0.0000767
P(7,0)	0.0000163	0.000016	0.0000186	0.0000176
P(7,1)	0.0001772	0.0001756	0.000012	0.0000109
P*(7,1)	0.0020054	0.0020045	0.0000225	0.0000199
P(8,0)	0.0000082	0.000008	0.0000046	0.0000042
P(8,1)	0.0000889	0.0000881	0.000003	0.0000027
P*(8,1)	0.0010065	0.001006	0.0000059	0.0000051
P(9,0)	0.0000041	0.000004	0.0000011	0.000001
P(9,1)	0.0000446	0.0000442	0.0000008	0.0000007
P*(9,1)	0.0005051	0.0005048	0.0000015	0.0000013
P(10,0)	0.0000021	0.000002	0.0000003	0.0000003
P(10,1)	0.0000224	0.0000222	0.0000002	0.0000002
P*(10,1)	0.0002535	0.0002533	0.0000004	0.0000003
P(11,0)	0.0000002	0.0000002	0.0000001	0.0000001
P(11,1)	0.0000017	0.0000017	0.0000001	0.0000001
P*(11,1)	0.0001364	0.0001362	0.0000001	0.0000001
P(12,0)	1E-08	0	0.0000001	0
Stable/unstable	Stable	Unstable	Stable	Unstable
Average length of queue in first stage	1.1305752	1.1296219	0.0576234	0.00553076
Average length of queue in system	0.2756027	0.2753201	0.6602436	0.6510576