The Tensor Products of Unital Quantales

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Abstract: In this paper, the definition of bimorphism of unital quantales is given, base on the tensor product of completed lattice, the concrete forms of the tensor product of unital quantales is obtained, and some properties of their are discussed.

Key words: Unital quantales; Unital quantales bimorphism; Tensor product; Quantale congruence


1. INTRODUCTION

Quantale was introduced by C.J.Mulvey in 1986 for studying the foundations of quantum logic and for studying non-commutation C*-algebras [1]. The systematic introduction of quantale theory came from the book [2], which written by K.I.Rosenthal in 1990. Quantale theory provides a powerful tool in studying non-commutative structures, it has a wide applications, especially in studying noncommutative C*-algebra theory [3], the ideal theory of commutative ring [4], linear logic [5] and so on. So, the quantale theory has aroused great interests of many researches [6–16].

Since tensor product is very important concept in many algebraic structures, and their tensor product have been studied systematically in [17–19]. In this paper, we
shall study the some properties of tensor product of unital quantales. For notions and concepts concerned, but explained, please refer to [2,20].

2. PRELIMINARIES

Definition 2.1 [2] A quantale is a complete lattice $Q$ with an associative binary operation “$\&$” satisfying:

$$a \& (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \& b_i) \quad \text{and} \quad (\bigvee_{i \in I} b_i) \& a = \bigvee_{i \in I} (b_i \& a),$$

for all $a, b_i \in Q$, where $I$ is a set, 0 and 1 denote the smallest element and the greatest element of $Q$, respectively.

A quantale $Q$ is said to be unital if there is an element $u \in Q$ such that $u \& a = a \& u = a$ for all $a \in Q$.

Definition 2.2 [2] Let $Q$ be a quantale and $a \in Q$.

(1) $a$ is right – sided if and only if $a \& 1 \leq a$.

(2) $a$ is left – sided if and only if $1 \& a \leq a$.

(3) $a$ is two – sided if and only if $a$ is both right and left side.

(4) $a$ is idempotent if and only if $a \& a = a$.

Definition 2.3 [2] Let $Q$ and $P$ be quantales. A function $f : Q \to P$ is a homomorphism of quantale if $f$ preserves arbitrary sups and the operation “$\&$”. If $Q$ and $P$ are unital, then $f$ is unital homomorphism if in addition to being a homomorphism, it satisfies $f(u_Q) = u_P$, where $u_Q$ and $u_P$ are units of $Q$ and $P$, respectively.

Definition 2.4 [2] Let $Q$ be a quantale. A subset $S \subseteq Q$ is a subquantale of $Q$ iff the inclusion $S \subseteq Q$ is a quantale homomorphism, i.e., $S$ is closed under sups and “$\&$”.

Take a set $X$ which is nonempty, $N$ is the set of natural number. Let $X^* = \{x_1x_2 \cdots x_n \mid x_i \in X, \ n \in N^+\}$. Define the operation “$*$” on $X$ such that

$$x_1x_2 \cdots x_n \ast y_1y_2 \cdots y_m = x_1x_2 \cdots x_ny_1y_2 \cdots y_m$$

for all $x_1x_2 \cdots x_n, y_1y_2 \cdots y_m \in X^*$.

Let $P(X^*)$ denote the set of all subset of $X^*$. Then $P(X^*)$ is clearly a complete lattice. Define the operation “$\&$” on $P(X^*)$ such that $A \& B = \{a \ast b \mid a \in A, \ b \in B\}$ for all $A, B \in P(X^*)$.

Theorem 2.5 [13] Let $X$ be a nonempty set. Then $(P(X^*), \&, \emptyset)$ be free unital quantales generated by $X$.

Definition 2.6 Let $Q_1, Q_2$ and $Q$ be quantales, a map $f : Q_1 \times Q_2 \to Q$ is said to be a bimorphism if it satisfying:

$$\forall \{x_i\}_{i \in I} \subseteq Q, \ \forall \{y_j\}_{j \in J} \subseteq Q_2, \ \forall x, s \in Q_1, \ \forall y, t \in Q_2,$$

(i) $f(\bigvee_{i \in I} x_i, y) = \bigvee_{i \in I} f(x_i, y)$;

(ii) $f(x, \bigvee_{j \in J} y_j) = \bigvee_{j \in J} f(x, y_j)$;

(iii) $f(x \& s, t) = f(x, t) \& f(s, t), \ f(x, y \& t) = f(x, y) \& f(x, t)$. 

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3. THE TENSOR PRODUCTS OF UNITAL QUANTALES

The present section is dedicated to the tensor product of unital quantales. We will show its existence, and some properties are discussed.

**Definition 3.1** Let $Q_1, Q_2$ and $Q_1 \otimes Q_2$ be unital quantales, the map $\tau : Q_1 \times Q_2 \rightarrow T$ be a bimorphism. We call $Q_1 \otimes Q_2$ be the tensor product of $Q_1$ and $Q_2$, such that for any unital quantales $Q$ and bimorphism $f : Q_1 \times Q_2 \rightarrow Q$, there exists a unique unital homomorphism $h_f : T \rightarrow Q$ with $h_f \circ \tau = f$, i.e. the triangle commutes.

$$
Q_1 \times Q_2 \xrightarrow{\tau} T \\
\downarrow{f} \quad \downarrow{h_f} \\
Q
$$

Obviously, by definition 3.1, if the tensor product of two unital quantales exists, then it is unique up to isomorphism.

**Theorem 3.2** Let $Q_1, Q_2$ be unital quantales, then tensor product $Q_1 \otimes Q_2$ exists, and it is up to isomorphisms, the quotient of the free unital quantales $P(\langle Q_1 \times Q_2 \rangle^*)$ with respect to the unital quantales congruence generated by the set

$$
R = \{(\{(x \& y_1, z\}, \{(x, z) \ast (y, z)\}, \{(x, y_2 \& z)\}, \{(x, y_2) \ast (x, z)\},
\bigcup_{A \subseteq Q_1} \{\bigvee A, y\}, \bigcup_{a \in A} \{(a, y)\}, \bigcup_{b \in B} \{(x, b)\},
\bigcup_{A \subseteq Q_1, B \subseteq Q_2} \{\bigvee A, y_{1, 2}, z\}
\bigcup_{a \in A} \mu(\bigvee A, y) = \bigcup_{a \in A} h_f(\bigcup \mu(a, y)) = \bigcup_{a \in A} h_f(\bigcup \{a\})$,}$$

Proof. Let $\mu : Q_1 \times Q_2 \rightarrow P(\langle Q_1 \times Q_2 \rangle^*)$ be the inclusion mapping, $f : Q_1 \times Q_2 \rightarrow Q$ be a unital quantales bimorphism. Since $P(\langle Q_1 \times Q_2 \rangle^*)$ be the free unital quantales generated by $Q_1 \times Q_2$, then there exists a unique unital quantales homomorphism $h_f : P(\langle Q_1 \times Q_2 \rangle^*) \rightarrow Q$ such that $h_f \circ \mu = f$ i.e. the triangle commutes

$$
Q_1 \times Q_2 \xrightarrow{\mu} P(\langle Q_1 \times Q_2 \rangle^*) \\
\downarrow{f} \quad \downarrow{h_f} \\
Q
$$

Since $f$ is a unital quantales bimorphism, we have that for all

$$
x, y_1 \in Q_1, y \in Q_2, A \subseteq Q_1, B \subseteq Q_2, y_2, z \in Q_2,$$

$$
h_f(\{(\bigvee A, y)\}) = h_f \circ \mu(\bigvee A, y) = h_f(\bigvee A, y) = \bigcup_{a \in A} f(A, y)$$

$$
= \bigcup_{a \in A} h_f(\bigcup_{a \in A} \mu(a, y)) = h_f(\bigcup_{a \in A} \{a\}),$$
i.e. 
\[ h_f(\bigvee A) = h_f(\bigcup_{a \in A} \{a, y\}) \]
and by symmetry
\[ h_f(\bigvee B) = h_f(\bigcup_{b \in B} \{a, b\}) \]

\[ h_f(\{(x \& y, z)\}) = h_f(\{(x \& y, z\}) = f(x \& y, z) = f(x, z) \& f(y, z) \]
\[ = (h_f(\mu(x, y, z)))(h_f(\mu(y, 1, z)) = h_f(\mu(x, 1, z)) = h_f(\{(x, z) \& (y, z)\}) \]

Thus, \( h_f(\{(x \& y, z)\}) = h_f(\{(x, z) \& (y, z)\}) \), and analogously
\[ h_f(\{(x, y, z)\}) = h_f(\{(x, y\}) \]

Therefore
\[ R \subseteq R_{h_f} = \{(x, y) \in P((Q_1 \times Q_2)^+) \times P((Q_1 \times Q_2)^+) \mid h_f(x) = h_f(y)\} \]

For convenience of expression, let \( T = P((Q_1 \times Q_2)^+) \) denote the quotient quantale of the free unital quantales with respect to the unital quantales congruence generated by the set \( R \). The mapping \( \tau : P((Q_1 \times Q_2)^+) \rightarrow T \) denotes the canonical epimorphism. We define the map \( h'_f : T \rightarrow Q \) such that \( h'_f([X]) = h_f(X) \) for all \([X] \in T\).

For all \([X] \in T, [Y] \in [X]\), since \((X, Y) \in \langle R \rangle \subseteq R_{h_f}\), then \( h_f(X) = h_f(Y)\). Hence \( h'_f \) is well defined.

Next, we will prove that is a unital quantales homomorphism, i.e. it preserves arbitrary join and operation \&.

For all \( \{[X_i]\}_{i \in I} \subseteq T, [X], [Y] \in T \), we have

(i) \[ h'_f(\bigvee_{i \in I} X_i) = h'_f(\bigcup_{i \in I} X_i) = h_f(\bigvee_{i \in I} X_i) = h_f(\bigcup_{i \in I} [X_i]) \]

(ii) \[ h'_f([X] \& [Y]) = h'_f([X \& Y]) = h_f(X \& Y) = h_f(X) \& h_f(Y) = h'_f([X]) \& h'_f([Y]) \]

\( \forall (x, y) \in Q_1 \times Q_2 \), then
\[ h'_f \circ \pi \circ \mu(x, y) = h'_f(\pi([x, y])) = h'_f([x, y]) \]
\[ = h_f([x, y]) = h_f(\mu(x, y)) = f(x, y) \]

i.e. \( h'_f \circ \pi \circ \mu = h_f \circ \mu = f \).

Let \( \tau = \pi \circ \mu \), for all \( \{x_i\}_{i \in I}, \{y_j\}_{j \in J} \subseteq Q_1, x \in Q_2, [t, s] \in Q_2 \), we can see that

(i) \[ \tau(\bigvee_{i \in I} x_i, y) = (\pi \circ \mu)(\bigvee_{i \in I} x_i, y) = \pi(\bigvee_{i \in I} x_i, y) \]
\[ \bigvee_{i \in I} [x, y] = \bigcup_{i \in I} [x_i, y], \text{ and} \]
\[ \bigvee_{i \in I} (\tau(x_i, y)) = \bigvee_{i \in I} (\pi \circ \mu)(x_i, y) = \bigvee_{i \in I} \pi(x_i, y) \]
\[ = \bigvee_{i \in I} [x_i, y] = \bigcup_{i \in I} [x_i, y]. \]
Since
\[\left(\bigvee_{i \in I} x_i \right) \times \{y\}, \bigcup_{i \in I} (x_i, y) \right) \in R \subseteq \langle R \rangle,\]
then
\[\bigcup_{i \in I} (x_i, y) = \bigcup_{i \in I} \left(\bigvee_{i \in I} x_i \right) \times \{y\}.\]
Hence
\[\tau\left(\bigvee_{i \in I} x_i, y\right) = \bigvee_{i \in I} \left(\tau(x_i, y)\right).\]
Similarly, we have
\[\tau(x, \bigvee_{j \in J} y_j) = \bigvee_{j \in J} \left(\tau(x, y_j)\right).\]

(ii) \(\tau(x \& s, t) = (\pi \circ \mu)(x \& s, t) = \pi(\{(x \& s, t)\})\)
\[= \pi(\{(x, t) \ast (s, t)\}) = \pi(\{(x, t)\} \& \{(s, t)\}) = \pi(\mu(x, t) \& \mu(s, t))\]
\[= (\pi \circ \mu(x, t)) \& (\pi \circ \mu(s, t)) = \tau(x, t) \& \tau(s, t).\]

Similarly, we have
\[\tau(x, y \& t) = \tau(x, y) \& \tau(x, t).\]

Therefore the map \(\tau\) is a unital quantales homomorphism.

It is easy to verify that \(h_f' \circ \tau = h_f' \circ \pi \circ \mu = h_f \circ \mu = f\), i.e. the diagram commute.

At last, we will prove that \(h_f\) is a unique quantale homomorphism.

Assume that \(h_f''\) is a unital quantale homomorphism such that \(h_f'' \circ \tau = f\).

For all \([X] \in T\), then \(h_f''([X]) = h_f'' \circ \pi(X) = h_f(X) = h_f' \circ \pi(X) = h_f'([X])\), i.e. \(h_f'' = h_f'.\)

Therefore, the tensor product \(Q_1 \otimes Q_2\) exists, and it is up to isomorphisms the quotient \(P((Q_1 \times Q_2)^*)/\langle R \rangle\).

\[\text{Definition 3.3} \quad \text{Let} \; Q_1, Q \text{ be unital quantale,} \; x \in Q_1, y \in Q_2. \text{ We denote by} \; x \otimes y \text{ the image of the pair} \; (x, y) \text{ under mapping} \; \tau, \text{i.e. the congruence class} \; [(x, y)] = x \otimes y, \text{ and we call} \; x \otimes y \text{ is tensor.} \]
**Theorem 3.4** Let $Q_1$ and $Q_2$ be unital quantales, then $Q_1 \otimes Q_2 = \{ \bigvee ((x_1 \otimes y_1) \& (x_2 \otimes y_2) \& \cdots (x_n \otimes y_n)) \mid x_n \in Q_1, y_n \in Q_2, n \in \mathbb{N}^+ \}$.

Let $Q_1, Q_2$ and $Q_3$ be unique quantales, we denote by $\text{Hom}(Q_1 \otimes Q_2, Q_3)$ the set of all unique quantale homomorphisms between $Q_1 \otimes Q_2$ and $Q_3$. $\text{Hom}_c(Q_1, \text{Hom}(Q_2, Q_3))$ the set of all complete lattice homomorphisms between $Q_1$ and $\text{Hom}(Q_2, Q_3)$.

Define $\eta : \text{Hom}(Q_1 \otimes Q_2, Q_3) \rightarrow \text{Hom}(Q_1, \text{Hom}(Q_2, Q_3))$

$$h \mapsto h : Q_1 \rightarrow \text{Hom}(Q_2, Q_3)$$

$$x \mapsto h_x : Q_2 \rightarrow Q_3$$

$$y \mapsto h(x \otimes y)$$

**Theorem 3.5** Let $Q_1, Q_2$ and $Q_3$ be unique quantales, then map $\eta$ be a complete lattice homomorphism.

**Proof.** Let $h : Q_1 \otimes Q_2 \rightarrow Q_3$ be a unique quantale homomorphism. For all $x \in Q$, define mapping $h_x : Q_2 \rightarrow Q_3$ such that $h_x(y) = h(x \otimes y)$ for all $y \in Q_2$. Next, we will prove that $h_x$ is a unique quantale homomorphism.

(i) $\forall \{y_i\}_{i \in I} \subseteq Q_2$, then

$$h_x(\bigvee_{i \in I} y_i) = h(x \otimes (\bigvee_{i \in I} y_i)) = h(\tau(x, \bigvee_{i \in I} y_i)) = h(\bigvee_{i \in I} \tau(x, y_i))$$

$$= h(\bigvee_{i \in I} y_i (x \otimes y_i)) = \bigvee_{i \in I} h(x \otimes y_i) = \bigvee_{i \in I} h_x(y_i);$$

(ii) $\forall y_1, y_2 \in Q_2$, then

$$h_x(y_1 \& y_2) = h(x \otimes (y_1 \& y_2)) = h(\tau(x, y_1 \& y_2)) = h(\tau(x, y_1) \&\tau(x, y_2)) = h(x \otimes y_1 \& h(x \otimes y_2) = h_x(y_1) \& h_x(y_2).$$

Therefore $h_x \in \text{Hom}(Q_1, Q_2)$ for all $x \in Q_1$.

Next, we will verify that $h : Q_1 \rightarrow \text{Hom}(Q_1, Q_2)$ is a complete lattice homomorphism, i.e. $h$ preserves arbitrary joins.

For all $\{x_i\}_{i \in I} \subseteq Q_1$, $y \in Q_2$, then

$$h(\bigvee_{i \in I} x_i, y) = h((\bigvee_{i \in I} x_i) \otimes y) = h(\tau(\bigvee_{i \in I} x_i, y))$$

$$= h(\bigvee_{i \in I} \tau(x_i, y)) = \bigvee_{i \in I} h(x_i \otimes y) = \bigvee_{i \in I} h_x(y).$$

For all $\{f^i\}_{i \in I} \subseteq \text{Hom}(Q_1 \otimes Q_2, Q_3)$, $x \in Q_1, y \in Q_2$, then

$$\eta(\bigvee_{i \in I} f^i)(x)(y) = (\bigvee_{i \in I} f^i)(x \otimes y) = \bigvee_{i \in I} (f^i(x \otimes y))$$

$$= \bigvee_{i \in I} f^i_x(y) = (\bigvee_{i \in I} f^i_x)(y) = \bigvee_{i \in I} \eta(f^i(x))(y).$$

Therefore, the map $\eta$ is well defined and is a complete lattice homomorphism. By theorem 3.4, we can see that the map $\eta$ is a injective.  

\[\square\]
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