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Oscillation of Nonlinear Delay Partial Difference Equations

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Abstract: In this paper, we consider certain nonlinear partial difference equations

$$aA_{m+1,n} + bA_{m,n+1} - cA_{m,n} + \sum_{i=1}^{u} p_i(m,n)A_{m-\sigma_i,n-\tau_i} = 0$$

where $a, b, c \in (0, \infty)$, u is a positive integer, $p_i(m,n)$, $(i = 0, 1, 2, \cdots u)$ are positive real sequences. $\sigma_i, \tau_i \in N_0 = \{1, 2, \cdots\}$, $i = 1, 2, \cdots, u$. A new comparison theorem for oscillation of the above equation is obtained.

Key words: Nonlinear partial; Difference equations; Eventually positive solutions

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1. INTRODUCTION

In this paper we consider nonlinear partial difference equation

$$aA_{m+1,n} + bA_{m,n+1} - cA_{m,n} + \sum_{i=1}^{u} p_i(m,n)A_{m-\sigma_i,n-\tau_i} = 0 \tag{1.1}$$

where $a, b, c \in (0, \infty)$, u is a positive integer, $p_i(m,n)$, $(i = 0, 1, 2, \cdots u)$ are positive real sequences. $\sigma_i, \tau_i \in N_0 = \{1, 2, \cdots\}$, $i = 1, 2, \cdots, u$. The purpose of this paper is to obtain a new comparison theorem for oscillation of all solutions of (1.1).

2. MAIN RESULTS

To prove our main result, we need several preparatory results.

Lemma 2.1. Assume that $\{A_{m,n}\}$ is a positive solution of (1.1). Then

$$\text{i}: \quad A_{m+1,n} \le \theta_1 A_{m,n}, \quad A_{m,n+1} \le \theta_2 A_{m,n}, \tag{2.1}$$

and

$$\text{ii}: \quad A_{m-\sigma_i,n-\tau_i} \ge \theta_1^{\sigma_0-\sigma_i}\theta_2^{\tau_0-\tau_i} A_{m-\sigma_0,n-\tau_0}, \tag{2.2}$$

where $\theta_1 = \dfrac{c}{a}$, $\theta_2 = \dfrac{c}{b}$, $\sigma_0 = \min\limits_{1\le i\le u}\{\sigma_i\}$, $\tau_0 = \min\limits_{1\le i\le u}\{\tau_i\}$.

Proof. Assume that $\{A_{m,n}\}$ is eventually positive solutions of (1.1). From (1.1), we have

$$aA_{m+1,n} + bA_{m,n+1} - cA_{m,n} = -\sum_{i=1}^{u} p_i(m,n)A_{m-\sigma_i,n-\tau_i} \le 0,$$

and so

$$aA_{m+1,n} + bA_{m,n+1} \le cA_{m,n}.$$

Hence $A_{m+1,n} \le \theta_1 A_{m,n}$ and $A_{m,n+1} \le \theta_2 A_{m,n}$. From the above inequality, we can find

$$A_{m,n} \le \theta_1^{\sigma_0} A_{m-\sigma_0,n} \le \theta_1^{\sigma_i} A_{m-\sigma_i,n}, \quad A_{m-\sigma_0,n} \le \theta_2^{\tau_0} A_{m-\sigma_0,n-\tau_0},$$

and

$$A_{m-\sigma_i,n} \le \theta_2^{\tau_0} A_{m-\sigma_i,n-\tau_0} \le \theta_2^{\tau_i} A_{m-\sigma_i,n-\tau_i}.$$

Hence

$$A_{m,n} \le \theta_1^{\sigma_0}\theta_2^{\tau_0} A_{m-\sigma_0,n-\tau_0} \le \theta_1^{\sigma_i}\theta_2^{\tau_i} A_{m-\sigma_i,n-\tau i}.$$

The proof of Lemma 2.1 is completed. □

Lemma 2.2. [1] If $x, y \in R^+$ and $x \ne y$, then

$$rx^{r-1}(x-y) > x^r - y^r > ry^{r-1}(x-y), \quad \text{for} \quad r > 1.$$

Theorem 2.1. If the difference inequality

$$aA_{m+1,n} + bA_{m,n+1} - cA_{m,n} + \sum_{i=1}^{u} \theta_1^{\sigma_0-\sigma_i}\theta_2^{\tau_0-\tau_i} p_i(m,n)A_{m-\sigma_0,n-\tau_0} \le 0 \tag{2.3}$$

has no eventually positive solutions, then every solution of equation (1.1) oscillates.

Proof. Assume that $\{A_{m,n}\}$ a is positive solution of equation (1.1). Then, by (1.1) and Lemma 2.2, we obtain

$$aA_{m+1,n} + bA_{m,n+1} - cA_{m,n} + \sum_{i=1}^{u} p_i(m,n)A_{m-\sigma_i,n-\tau_i} \le 0 \tag{2.4}$$

Substituting (2.2) into (2.4), we have

$$aA_{m+1,n} + bA_{m,n+1} - cA_{m,n} + \sum_{i=1}^{u} \theta_1^{\sigma_0-\sigma_i}\theta_2^{\tau_0-\tau_i} p_i(m,n)A_{m-\sigma_0,n-\tau_0} \le 0.$$

This contradiction completes the proof. □

Define a set E by

$$E = \{\lambda > 0 | c - \lambda Q_{m,n} > 0, \text{eventually}\}$$

where $Q_{m,n} = \sum_{i=1}^{u} \theta_1^{\sigma_0-\sigma_i} \theta_2^{\tau_0-\tau_i} p_i(m,n)$.

Theorem 2.2. Assume that

(i) $\lim_{m,n\to\infty} \sup Q_{m,n} > 0$;

(ii) there exists $M \geq m_0$, $N \geq n_0$ such that if $\sigma_0 > \tau_0 > 0$,

$$\sup_{\lambda\in E, M\geq m, N\geq n} \lambda \left[\prod_{j=1}^{\sigma_0-\tau_0} \prod_{i=1}^{\tau_0} (c - \lambda Q_{m-i-j,n-i}) \right]^{\frac{1}{\sigma_0-\tau_0}} < \left(\frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\tau_0} \theta_1^{\tau_0-\sigma_0}, \tag{2.5}$$

and if $\tau_0 > \sigma_0 > 0$,

$$\sup_{\lambda\in E, M\geq m, N\geq n} \lambda \left[\prod_{j=1}^{\tau_0-\sigma_0} \prod_{i=1}^{\sigma_0} (c - \lambda Q_{m-i,n-i-j}) \right]^{\frac{1}{\tau_0-\sigma_0}} < \left(\frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\sigma_0} \theta_2^{\sigma_0-\tau_0}. \tag{2.6}$$

Then every solution of (1.1) oscillates.

Proof. Suppose, to the contrary, $A_{m,n}$ is an eventually positive solution. We define a subset S of the positive numbers as follows:

$$S(\lambda) = \{\lambda > 0 | aA_{m+1,n} + bA_{m,n+1} - [c - \lambda Q_{m,n}]A_{m,n} \leq 0, \quad \text{eventually}\}.$$

From (2.3) and Lemma 2.1, we have

$$aA_{m+1,n} + bA_{m,n+1} - (c - \theta_1^{-\sigma_0}\theta_2^{-\tau_0} Q_{m,n})A_{m,n} \leq 0,$$

which implies $\theta_1^{-\sigma_0}\theta_2^{-\tau_0} \in S(\lambda)$. Hence, $S(\lambda)$ is nonempty. For $\lambda \in S$, we have eventually that $c - \lambda Q_{m,n} > 0$, which implies that $S \subset E$, Due to condition (i), the set E is bounded, and hence, $S(\lambda)$ is bounded. Let $u \in S$. Then from Lemma 2.1, we have

$$\begin{aligned} \left(\frac{a}{\theta_2} + \frac{b}{\theta_1} \right) A_{m+1,n+1} &\leq aA_{m+1,n} + bA_{m,n+1} \\ &\leq (c - uQ_{m,n})A_{m,n}. \end{aligned}$$

If $\sigma_0 > \tau_0 > 0$, then

$$A_{m,n} \leq \left(\frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{-\tau_0} \prod_{i=1}^{\tau_0} (c - uQ_{m-i,n-i}) A_{m-\tau_0,n-\tau_0},$$

and for $j = 1, 2, \cdots, \sigma_0 - \tau_0$, we have

$$\begin{aligned} A_{m-j,n} &\leq \left(\frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{-\tau_0} \prod_{i=1}^{\tau_0} (c - uQ_{m-i-j,n-i}) A_{m-\tau_0-j,n-\tau_0} \\ &\leq \theta_1^{\sigma_0-\tau_0-j} \left(\frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{-\tau_0} \prod_{i=1}^{\tau_0} (c - uQ_{m-i-j,n-i}) A_{m-\sigma_0,n-\tau_0}. \end{aligned} \tag{2.7}$$

Now, from Lemma 2.1 and (2.7), it follows that

$$A_{m,n}^{\sigma_0-\tau_0} \leq \left(\frac{a}{\theta_2}+\frac{b}{\theta_1}\right)^{-\tau_0(\sigma_0-\tau_0)} \theta_1^{(\sigma_0-\tau_0)^2} \left[\prod_{j=1}^{\sigma_0-\tau_0}\prod_{i=1}^{\tau_0}(c-uQ_{m-i-j,n-i})\right] A_{m-\sigma_0,n-\tau_0}^{\sigma_0-\tau_0},$$

i.e.,

$$A_{m,n} \leq \left\{\left(\frac{a}{\theta_2}+\frac{b}{\theta_1}\right)^{-\tau_0(\sigma_0-\tau_0)} \theta_1^{(\sigma_0-\tau_0)^2} \left[\prod_{j=1}^{\sigma_0-\tau_0}\prod_{i=1}^{\tau_0}(c-uQ_{m-i-j,n-i})\right]\right\}^{\frac{1}{\sigma_0-\tau_0}} A_{m-\sigma_0,n-\tau_0}. \tag{2.8}$$

Similarly, if $\tau_0 > \sigma_0 > 0$, then

$$A_{m,n} \leq \left\{\left(\frac{a}{\theta_2}+\frac{b}{\theta_1}\right)^{-\sigma_0(\tau_0-\sigma_0)} \theta_2^{(\tau_0-\sigma_0)^2} \left[\prod_{j=1}^{\tau_0-\sigma_0}\prod_{i=1}^{\tau_0}(c-uQ_{m-i,n-i-j})\right]\right\}^{\frac{1}{\tau_0-\sigma_0}} A_{m-\sigma_0,n-\tau_0}. \tag{2.9}$$

Substituting (2.8) and (2.9) into (2.3), we get respectively, for $\sigma_0 > \tau_0$,

$$aA_{m+1,n}+bA_{m,n+1}-cA_{m,n}$$

$$+Q_{m,n}\left(\frac{a}{\theta_2}+\frac{b}{\theta_1}\right)^{\tau_0} \theta_1^{\tau_0-\sigma_0} \left[\prod_{j=1}^{\sigma_0-\tau_0}\prod_{i=1}^{\tau_0}(c-uQ_{m-i-j,n-i})\right]^{\frac{1}{\tau_0-\sigma_0}} A_{m,n} \leq 0,$$

and for $\tau_0 > \sigma_0$,

$$aA_{m+1,n}+bA_{m,n+1}-cA_{m,n}$$

$$+Q_{m,n}\left(\frac{a}{\theta_2}+\frac{b}{\theta_1}\right)^{\sigma_0} \theta_2^{\sigma_0-\tau_0} \left[\prod_{j=1}^{\tau_0-\sigma_0}\prod_{i=1}^{\sigma_0}(c-uQ_{m-i,n-i-j})\right]^{\frac{1}{\sigma_0-\tau_0}} A_{m,n} \leq 0.$$

Hence, for $\sigma_0 > \tau_0$,

$$\begin{aligned} &aA_{m+1,n}+bA_{m,n+1}-\Big\{c-Q_{m,n}\left(\frac{a}{\theta_2}+\frac{b}{\theta_1}\right)^{\tau_0} \theta_1^{\tau_0-\sigma_0} \\ &\times \sup_{m\geq M,n\geq N}\left[\prod_{j=1}^{\sigma_0-\tau_0}\prod_{i=1}^{\tau_0}(c-uQ_{m-i-j,n-i})\right]^{\frac{1}{\tau_0-\sigma_0}}\Big\}A_{m,n} \leq 0, \end{aligned} \tag{2.10}$$

and for $\tau_0 > \sigma_0$,

$$aA_{m+1,n} + bA_{m,n+1} - \Big\{c - Q_{m,n}\left(\frac{a}{\theta_2}+\frac{b}{\theta_1}\right)^{\sigma_0}\theta_2^{\sigma_0-\tau_0} \times \sup_{m\geq M,n\geq N}\left[\prod_{j=1}^{\tau_0-\sigma_0}\prod_{i=1}^{\sigma_0}(c-uQ_{m-i,n-i-j})\right]^{\frac{1}{\sigma_0-\tau_0}}\Big\}A_{m,n} \leq 0. \tag{2.11}$$

From (2.10) and (2.11), we get

$$\left(\frac{a}{\theta_2}+\frac{b}{\theta_1}\right)^{\tau_0}\theta_1^{\tau_0-\sigma_0}\left\{\sup_{m\geq M,n\geq N}\left[\prod_{j=1}^{\sigma_0-\tau_0}\prod_{i=1}^{\tau_0}(c-uQ_{m-i-j,n-i})\right]^{\frac{1}{\tau_0-\sigma_0}}\right\}\in S \quad \text{for } \sigma_0 > \tau_0, \tag{2.12}$$

and

$$\left(\frac{a}{\theta_2}+\frac{b}{\theta_1}\right)^{\sigma_0}\theta_2^{\sigma_0-\tau_0}\left(\sup_{m\geq M,n\geq N}\left[\prod_{j=1}^{\tau_0-\sigma_0}\prod_{i=1}^{\sigma_0}(c-uQ_{m-i,n-i-j})\right]^{\frac{1}{\sigma_0-\tau_0}}\right)\in S \quad \text{for } \tau_0 > \sigma_0. \tag{2.13}$$

On the other hand, (2.5) implies that there exists $a_1 \in (0,1)$ (we can choose the same) such that for $\sigma_0 > \tau_0$

$$\sup_{\lambda\in E,m\geq M,n\geq N}\lambda\left[\prod_{j=1}^{\sigma_0-\tau_0}\prod_{i=1}^{\tau_0}(c-\lambda Q_{m-i-j,n-i})\right]^{\frac{1}{\sigma_0-\tau_0}} \leq a_1\left(\frac{a}{\theta_2}+\frac{b}{\theta_1}\right)^{\tau_0}\theta_1^{\tau_0-\sigma_0}, \tag{2.14}$$

and (2.6) implies that there exists $a_1 \in (0,1)$ (we can choose the same) such that for $\tau_0 > \sigma_0 > 0$,

$$\sup_{\lambda\in E,M\geq m,N\geq n}\lambda\left[\prod_{j=1}^{\tau_0-\sigma_0}\prod_{i=1}^{\sigma_0}(c-\lambda Q_{m-i,n-i-j})\right]^{\frac{1}{\tau_0-\sigma_0}} \leq a_1\left(\frac{a}{\theta_2}+\frac{b}{\theta_1}\right)^{\sigma_0}\theta_2^{\sigma_0-\tau_0}. \tag{2.15}$$

In particular, (2.14) and (2.15) lead to (when $\lambda = u$), respectively,

$$\left(\frac{a}{\theta_2}+\frac{b}{\theta_1}\right)^{\tau_0}\theta_1^{\tau_0-\sigma_0}\sup_{\lambda\in E,m\geq M,n\geq N}\left[\prod_{j=1}^{\sigma_0-\tau_0}\prod_{i=1}^{\tau_0}(c-uQ_{m-i-j,n-i})\right]^{\frac{1}{\tau_0-\sigma_0}} \geq \frac{u}{a_1} \quad \text{for } \sigma_0 > \tau_0, \tag{2.16}$$

and

$$\left(\frac{a}{\theta_2}+\frac{b}{\theta_1}\right)^{\sigma_0}\theta_2^{\sigma_0-\tau_0}\sup_{\lambda\in E, M\geq m, N\geq n}\left[\prod_{j=1}^{\tau_0-\sigma_0}\prod_{i=1}^{\sigma_0}(c-uQ_{m-i,n-i-j})\right]^{\dfrac{1}{\sigma_0-\tau_0}}\geq\frac{u}{a_1}$$
$$\text{for}\quad \tau_0>\sigma_0. \tag{2.17}$$

Since $u\in S$ and $u'\leq u$ implies that $u'\in S$, it follows from (2.12) and (2.16) for $\sigma_0>\tau_0$, (2.13) and (2.17) for $\tau_0>\sigma_0$ that $\frac{u}{a_1}\in S$. Repeating the above arguments with u replaced by $\frac{u}{a_1}$, we get $\frac{u}{a_1a_2}\in S$, where $a_2\in(0,1)$. Continuing in this way, we obtain $\frac{u}{\prod_{i=1}^{\infty}a_i}\in S$, where $a_i\in(0,1)$. This contradicts the boundedness of S. The proof is complete. □

Corollary 2.1. In addition to (i) of Theorem 2.1, assume that for $\sigma_0>\tau_0>0$,

$$\lim_{m,n\to\infty}\inf\frac{1}{(\sigma_0-\tau_0)\tau_0}\sum_{j=1}^{\sigma_0-\tau_0}\sum_{i=1}^{\tau_0}Q_{m-i-j,n-i}>\frac{c^{\tau_0+1}\tau_0^{\tau_0}}{(\tau_0+1)^{\tau_0+1}}\left(\frac{a}{\theta_2}+\frac{b}{\theta_1}\right)^{-\tau_0}\theta_1^{\sigma_0-\tau_0},$$

and for $\tau_0>\sigma_0>0$,

$$\lim_{m,n\to\infty}\inf\frac{1}{(\tau_0-\sigma_0)\sigma_0}\sum_{j=1}^{\tau_0-\sigma_0}\sum_{i=1}^{\sigma_0}Q_{m-i,n-i-j}>\frac{c^{\sigma_0+1}\sigma_0^{\sigma_0}}{(\sigma_0+1)^{\sigma_0+1}}\left(\frac{a}{\theta_2}+\frac{b}{\theta_1}\right)^{-\sigma_0}\theta_2^{\tau_0-\sigma_0}.$$

Then every solution of (1.1) oscillates.

Proof. We note that

$$\max_{\frac{c}{e}>\lambda>0}\lambda(c-\lambda e)^{\tau_0}=\frac{c^{\tau_0+1}\tau_0^{\tau_0}}{e(\tau_0+1)^{\tau_0+1}}.$$

We shall use this for

$$e=\frac{1}{(\sigma_0-\tau_0)\tau_0}\sum_{j=1}^{\sigma_0-\tau_0}\sum_{i=1}^{\tau_0}Q_{m-i-j,n-i}.$$

Clearly,

$$\begin{aligned}&\lambda[\prod_{j=1}^{\sigma_0-\tau_0}\prod_{i=1}^{\tau_0}(c-\lambda Q_{m-i-j,n-i})]^{\frac{1}{\sigma_0-\tau_0}}\\ \leq\ &\lambda[\frac{1}{(\sigma_0-\tau_0)\tau_0}\sum_{j=1}^{\sigma_0-\tau_0}\sum_{i=1}^{\tau_0}(c-\lambda Q_{m-i-j,n-i})]^{\tau_0}\\ \leq\ &\lambda[c-\frac{\lambda}{(\sigma_0-\tau_0)\tau_0}\sum_{j=1}^{\sigma_0-\tau_0}\sum_{i=1}^{\tau_0}(Q_{m-i-j,n-i})]^{\tau_0}\end{aligned}$$

$$\begin{aligned} &\leq \frac{c^{\tau_0+1}\tau_0^{\tau_0}}{e(\tau_0+1)^{\tau_0+1}} \\ &< (\frac{a}{\theta_2}+\frac{b}{\theta_1})^{\tau_0}\theta_1^{\tau_0-\sigma_0}. \end{aligned}$$

Similarly, we have

$$\lambda\left[\prod_{j=1}^{\tau_0-\sigma_0}\prod_{i=1}^{\sigma_0}(c-\lambda Q_{m-i,n-i-j})\right]^{\frac{1}{\tau_0-\sigma_0}} < \left(\frac{a}{\theta_2}+\frac{b}{\theta_1}\right)^{\sigma_0}\theta_2^{\sigma_0-\tau_0}.$$

By Theorem 2.1, every solutions of (1.1) oscillates. The proof is complete. □

By a similar argument, we have the following results:

Corollary 2.2. If the condition of Theorem 2.2 holds, and

$$\lim_{m,n\to\infty}\inf Q_{m,n} = q > \frac{c^{\sigma_0+1}\sigma_0^{\sigma_0}}{(\sigma_0+1)^{\sigma_0+1}}\left(\frac{a}{\theta_2}+\frac{b}{\theta_1}\right)^{-\sigma_0},$$

then every solution of (1.1) oscillates.

Theorem 2.3. Assume that

(i) $\lim_{m,n\to\infty}\sup Q_{m,n} > 0$;

(ii) for $\sigma_0, \tau_0 > 0$,

$$\lim_{m,n\to\infty}\inf Q_{m,n} = q > 0, \tag{2.20}$$

and

$$\lim_{m,n\to\infty} Q_{m,n} > c\theta_1^{\sigma_0}\theta_2^{\tau_0} - \frac{a\theta_1+b\theta_2}{c}q > 0. \tag{2.21}$$

Then every solution of (1.1) oscillates.

Proof. Suppose, to the contrary, $A_{m,n}$ is an eventually positive solution. From (2.3) and (2.20), for any $\epsilon > 0$, we have $Q_{m,n} > q-\epsilon$ for $m \geq M, n \geq N$. From (2.3), Lemma 2.1 and above inequality, we obtain

$$A_{m,n} \geq \frac{(q-\epsilon)}{c}A_{m-\sigma_0,n-\tau_0} \geq \frac{(q-\epsilon)}{c}\theta_1^{1-\sigma_0}\theta_2^{1-\tau_0}A_{m-1,n-1},$$

$$A_{m,n} \geq \frac{(q-\epsilon)}{c}\theta_1^{1-\sigma_0}\theta_2^{-\tau_0}A_{m-1,n}, \quad \text{and} \quad A_{m,n} \geq \frac{(q-\epsilon)}{c}\theta_1^{-\sigma_0}\theta_2^{1-\tau_0}A_{m,n-1}.$$

Substituting above inequalities into (2.3), we get

$$\left[\frac{a\theta_1^{1-\sigma_0}\theta_2^{-\tau_0}+b\theta_1^{-\sigma_0}\theta_2^{1-\tau_0}}{c}(q-\epsilon)-c+Q_{m,n}\theta_1^{-\sigma_0}\theta_2^{-\tau_0}\right]A_{m,n} < 0,$$

which implies

$$\lim_{m,n\to\infty} Q_{m,n} \leq c\theta_1^{\sigma_0}\theta_2^{\tau_0} - \frac{a\theta_1+b\theta_2}{c}q > 0.$$

This contradicts (2.21). The proof is complete. □

Theorem 2.4. Assume that
(i) $\lim_{m,n\to\infty} \sup Q_{m,n} > 0$;
(ii) $\sigma_0 = \tau_0 = 0$, and

$$\lim_{m,n\to\infty} \sup Q_{m,n} > c. \qquad (2.22).$$

Then every solution of (1.1) oscillates.

Proof. Let $u \in S$. Then from (2.3) and Lemma 2.1, we have $-c + Q_{m,n}A_{m,n} < 0$, which implies $\lim_{m,n\to\infty} \sup Q_{m,n} \leq c$. This contradicts (2.22). The proof is complete. □

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