Oscillation of Nonlinear Delay Partial Difference Equations

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Abstract: In this paper, we consider certain nonlinear partial difference equations

\[ aA_{m+1,n} + bA_{m,n+1} - cA_{m,n} + \sum_{i=1}^{u} p_i(m,n)A_{m-\sigma_i,n-\tau_i} = 0 \]

where \( a, b, c \in (0, \infty) \), \( u \) is a positive integer, \( p_i(m,n), (i = 0, 1, 2, \cdots u) \) are positive real sequences. \( \sigma_i, \tau_i \in N_0 = \{1, 2, \cdots\}, i = 1, 2, \cdots, u \). A new comparison theorem for oscillation of the above equation is obtained.

Key words: Nonlinear partial; Difference equations; Eventually positive solutions

1. INTRODUCTION

In this paper we consider nonlinear partial difference equation

\[ aA_{m+1,n} + bA_{m,n+1} - cA_{m,n} + \sum_{i=1}^{u} p_i(m,n)A_{m-\sigma_i,n-\tau_i} = 0 \]  

where \( a, b, c \in (0, \infty) \), \( u \) is a positive integer, \( p_i(m,n), (i = 0, 1, 2, \cdots u) \) are positive real sequences. \( \sigma_i, \tau_i \in N_0 = \{1, 2, \cdots\}, i = 1, 2, \cdots, u \). The purpose of this paper is to obtain a new comparison theorem for oscillation of all solutions of (1.1).
2. MAIN RESULTS

To prove our main result, we need several preparatory results.

**Lemma 2.1.** Assume that \( \{A_{m,n}\} \) is a positive solution of (1.1). Then

\[
i: \quad A_{m+1,n} \leq \theta_1 A_{m,n}, \quad A_{m,n+1} \leq \theta_2 A_{m,n},
\]

and

\[
ii: \quad A_{m-\sigma, n-\tau} \geq \theta_1^{\sigma_0-\sigma_i} \theta_2^{\tau_0-\tau_i} A_{m-\sigma_0,n-\tau_0},
\]

where \( \theta_1 = \frac{c}{a}, \quad \theta_2 = \frac{c}{b}, \quad \sigma_0 = \min_{1 \leq i \leq u} \{\sigma_i\}, \quad \tau_0 = \min_{1 \leq i \leq u} \{\tau_i\}. \)

**Proof.** Assume that \( \{A_{m,n}\} \) is eventually positive solutions of (1.1). From (1.1), we have

\[
aA_{m+1,n} + bA_{m,n+1} - cA_{m,n} = -\sum_{i=1}^{u} p_i(m,n) A_{m-\sigma_i,n-\tau_i} \leq 0,
\]

and so

\[
aA_{m+1,n} + bA_{m,n+1} \leq cA_{m,n}.
\]

Hence \( A_{m+1,n} \leq \theta_1 A_{m,n} \) and \( A_{m,n+1} \leq \theta_2 A_{m,n} \). From the above inequality, we can find

\[
A_{m,n} \leq \theta_1^{\sigma_0} A_{m-\sigma_0,n} \leq \theta_1^{\sigma_i} A_{m-\sigma_i,n}, \quad A_{m-\sigma_0,n} \leq \theta_2^{\tau_0} A_{m-\sigma_0,n-\tau_0},
\]

and

\[
A_{m-\sigma_i,n} \leq \theta_2^{\tau_0} A_{m-\sigma_i,n-\tau_0} \leq \theta_2^{\tau_i} A_{m-\sigma_i,n-\tau_i}.
\]

Hence

\[
A_{m,n} \leq \theta_1^{\sigma_0} \theta_2^{\tau_0} A_{m-\sigma_0,n-\tau_0} \leq \theta_1^{\sigma_i} \theta_2^{\tau_i} A_{m-\sigma_i,n-\tau_i}.
\]

The proof of Lemma 2.1 is completed. \( \square \)

**Lemma 2.2.** \([1]\) If \( x, y \in \mathbb{R}^+ \) and \( x \neq y \), then

\[
rx^{r-1}(x - y) > x^r - y^r > ry^{r-1}(x - y), \quad \text{for} \quad r > 1.
\]

**Theorem 2.1.** If the difference inequality

\[
aA_{m+1,n} + bA_{m,n+1} - cA_{m,n} + \sum_{i=1}^{u} \theta_1^{\sigma_0-\sigma_i} \theta_2^{\tau_0-\tau_i} p_i(m,n) A_{m-\sigma_0,n-\tau_0} \leq 0
\]

has no eventually positive solutions, then every solution of equation (1.1) oscillates.

**Proof.** Assume that \( \{A_{m,n}\} \) is a positive solution of equation (1.1). Then, by (1.1) and Lemma 2.2, we obtain

\[
aA_{m+1,n} + bA_{m,n+1} - cA_{m,n} + \sum_{i=1}^{u} p_i(m,n) A_{m-\sigma_i,n-\tau_i} \leq 0
\]

(2.4)

Substituting (2.2) into (2.4), we have

\[
aA_{m+1,n} + bA_{m,n+1} - cA_{m,n} + \sum_{i=1}^{u} \theta_1^{\sigma_0-\sigma_i} \theta_2^{\tau_0-\tau_i} p_i(m,n) A_{m-\sigma_0,n-\tau_0} \leq 0.
\]

This contradiction completes the proof. \( \square \)
Define a set $E$ by

$$E = \{ \lambda > 0 | e - \lambda Q_{m,n} > 0, \text{eventually} \}$$

where $Q_{m,n} = \sum_{i=1}^{u} \theta_{1}^{\sigma_{0} - \sigma_{i}} \theta_{2}^{\tau_{0} - \tau_{i}} p_{i}(m,n)$.

**Theorem 2.2.** Assume that

(i) $\lim_{m,n \to \infty} \sup Q_{m,n} > 0$;

(ii) there exists $M \geq m_0$, $N \geq n_0$ such that if $\sigma_{0} > \tau_{0} > 0$,

$$\sup_{\lambda \in E, M \geq m, N \geq n} \lambda \left[ \prod_{j=1}^{r_{0}} \prod_{i=1}^{\sigma_{0}} (c - \lambda Q_{m-i,j,n-i}) \right] \frac{1}{\sigma_{0} - \tau_{0}} < \left( \frac{a}{\theta_{2}} + \frac{b}{\theta_{1}} \right)^{-\sigma_{0} + \tau_{0}} \theta_{1}^{\tau_{0} - \sigma_{0}},$$

and if $\tau_{0} > \sigma_{0} > 0$,

$$\sup_{\lambda \in E, M \geq m, N \geq n} \lambda \left[ \prod_{j=1}^{r_{0}} \prod_{i=1}^{\sigma_{0}} (c - \lambda Q_{m-i,j,n-i}) \right] \frac{1}{\tau_{0} - \sigma_{0}} < \left( \frac{a}{\theta_{2}} + \frac{b}{\theta_{1}} \right)^{\sigma_{0} - \tau_{0}} \theta_{2}^{\sigma_{0} - \tau_{0}}.$$

Then every solution of (1.1) oscillates.

**Proof.** Suppose, to the contrary, $A_{m,n}$ is an eventually positive solution. We define a subset $S$ of the positive numbers as follows:

$$S(\lambda) = \{ \lambda > 0 | aA_{m+1,n} + bA_{m,n+1} - [c - \lambda Q_{m,n}]A_{m,n} \leq 0, \text{ eventually} \}.$$

From (2.3) and Lemma 2.1, we have

$$aA_{m+1,n} + bA_{m,n+1} - (c - \theta_{1}^{-\sigma_{0}} \theta_{2}^{-\tau_{0}} Q_{m,n})A_{m,n} \leq 0,$$

which implies $\theta_{1}^{-\sigma_{0}} \theta_{2}^{-\tau_{0}} \in S(\lambda)$. Hence, $S(\lambda)$ is nonempty. For $\lambda \in S$, we have eventually that $c - \lambda Q_{m,n} > 0$, which implies that $S \subset E$. Due to condition (i), the set $E$ is bounded, and hence, $S(\lambda)$ is bounded. Let $u \in S$. Then from Lemma 2.1, we have

$$\left( \frac{a}{\theta_{2}} + \frac{b}{\theta_{1}} \right) A_{m+1,n+1} \leq aA_{m+1,n} + bA_{m,n+1} \leq (c - uQ_{m,n})A_{m,n}.$$

If $\sigma_{0} > \tau_{0} > 0$, then

$$A_{m,n} \leq \left( \frac{a}{\theta_{2}} + \frac{b}{\theta_{1}} \right)^{-\tau_{0} \sum_{i=1}^{\sigma_{0}} (c - uQ_{m-i,n-i})A_{m-\tau_{0},n-\tau_{0}},$$

and for $j = 1, 2, \cdots, \sigma_{0} - \tau_{0}$, we have

$$A_{m-j,n} \leq \left( \frac{a}{\theta_{2}} + \frac{b}{\theta_{1}} \right)^{-\tau_{0}} \prod_{i=1}^{\sigma_{0} - \tau_{0}} (c - uQ_{m-i-j,n-i})A_{m-\tau_{0}-j,n-\tau_{0}} \leq \theta_{1}^{-\sigma_{0} - \tau_{0} - j} \left( \frac{a}{\theta_{2}} + \frac{b}{\theta_{1}} \right)^{-\tau_{0}} \prod_{i=1}^{\sigma_{0} - \tau_{0}} (c - uQ_{m-i-j,n-i})A_{m-\sigma_{0},n-\tau_{0}}.$$
Now, from Lemma 2.1 and (2.7), it follows that

\[
A_{m,n}^{\sigma_0-\tau_0} \leq \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{-\tau_0(\sigma_0-\tau_0)} \theta_1^{(\sigma_0-\tau_0)^2} \left[ \prod_{j=1}^{\sigma_0-\tau_0} \prod_{i=1}^{\tau_0} (c - uQ_{m-i-j,n-i}) \right] A_{m-\sigma_0,n-\tau_0},
\]

i.e.,

\[
A_{m,n} \leq \left\{ \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{-\tau_0(\sigma_0-\tau_0)} \theta_1^{(\sigma_0-\tau_0)^2} \left[ \prod_{j=1}^{\sigma_0-\tau_0} \prod_{i=1}^{\tau_0} (c - uQ_{m-i-j,n-i}) \right] \right\}^{1/(\sigma_0-\tau_0)} A_{m-\sigma_0,n-\tau_0}.
\]

(2.8)

Similarly, if \( \tau_0 > \sigma_0 > 0 \), then

\[
A_{m,n} \leq \left\{ \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{-\sigma_0(\tau_0-\sigma_0)} \theta_2^{(\tau_0-\sigma_0)^2} \left[ \prod_{j=1}^{\tau_0-\sigma_0} \prod_{i=1}^{\sigma_0} (c - uQ_{m-i-j,n-i-j}) \right] \right\}^{1/(\tau_0-\sigma_0)} A_{m-\sigma_0,n-\tau_0}.
\]

(2.9)

Substituting (2.8) and (2.9) into (2.3), we get respectively, for \( \sigma_0 > \tau_0 \),

\[
aA_{m+1,n} + bA_{m,n+1} - cA_{m,n}
\]

\[
+ Q_{m,n} \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\tau_0} \theta_1^{\tau_0-\sigma_0} \left[ \prod_{j=1}^{\tau_0-\sigma_0} \prod_{i=1}^{\sigma_0} (c - uQ_{m-i-j,n-i}) \right] \frac{1}{\tau_0 - \sigma_0} A_{m,n} \leq 0,
\]

and for \( \tau_0 > \sigma_0 \),

\[
aA_{m+1,n} + bA_{m,n+1} - cA_{m,n}
\]

\[
+ Q_{m,n} \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\sigma_0} \theta_2^{\sigma_0-\tau_0} \left[ \prod_{j=1}^{\sigma_0-\tau_0} \prod_{i=1}^{\tau_0} (c - uQ_{m-i-j,n-i-j}) \right] \frac{1}{\sigma_0 - \tau_0} A_{m,n} \leq 0.
\]

Hence, for \( \sigma_0 > \tau_0 \),

\[
aA_{m+1,n} + bA_{m,n+1} - \left\{ c - Q_{m,n} \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\tau_0} \theta_1^{\tau_0-\sigma_0} \right\}^\ast \left( \prod_{j=1}^{\tau_0-\sigma_0} \prod_{i=1}^{\sigma_0} (c - uQ_{m-i-j,n-i}) \right) \frac{1}{\tau_0 - \sigma_0} \}
\]

(2.10)
and for $\tau_0 > \sigma_0$, \[
A_{m+1,n} + bA_{m,n+1} - \{ c - Q_{m,n} \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right) \theta_2^{\tau_0 - \tau_0} \}
\times \sup_{m \geq M, n \geq N} \left[ \prod_{j=1}^{\tau_0 - \sigma_0} \prod_{i=1}^{\sigma_0} (c - uQ_{m-i,n-i-j}) \right] \frac{1}{\tau_0 - \tau_0} A_{m,n} \leq 0. \tag{2.11}
\]

From (2.10) and (2.11), we get \[
\left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\tau_0} \theta_1^{\tau_0 - \sigma_0} \left\{ \sup_{m \geq M, n \geq N} \left[ \prod_{j=1}^{\tau_0 - \sigma_0} \prod_{i=1}^{\tau_0} (c - uQ_{m-i,n-i-j}) \right] \frac{1}{\tau_0 - \sigma_0} \right\} \in S
\text{ for } \sigma_0 > \tau_0, \tag{2.12}
\]

and \[
\left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\sigma_0} \theta_2^{\sigma_0 - \tau_0} \left\{ \sup_{m \geq M, n \geq N} \left[ \prod_{j=1}^{\tau_0 - \sigma_0} \prod_{i=1}^{\tau_0} (c - uQ_{m-i,n-i-j}) \right] \frac{1}{\sigma_0 - \tau_0} \right\} \in S
\text{ for } \tau_0 > \sigma_0. \tag{2.13}
\]

On the other hand, (2.5) implies that there exists $a_1 \in (0, 1)$ (we can choose the same) such that for $\tau_0 > \sigma_0$ \[
\sup_{\lambda \in E, m \geq M, n \geq N} \lambda \left[ \prod_{j=1}^{\tau_0 - \sigma_0} \prod_{i=1}^{\tau_0} (c - \lambda Q_{m-i,n-i-j}) \right] \frac{1}{\tau_0 - \tau_0} \leq a_1 \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\tau_0} \theta_1^{\tau_0 - \sigma_0}, \tag{2.14}
\]

and (2.6) implies that there exists $a_1 \in (0, 1)$ (we can choose the same) such that for $\tau_0 > \sigma_0 > 0$, \[
\sup_{\lambda \in E, M \geq m, N \geq n} \lambda \left[ \prod_{j=1}^{\tau_0 - \sigma_0} \prod_{i=1}^{\sigma_0} (c - \lambda Q_{m-i,n-i-j}) \right] \frac{1}{\tau_0 - \tau_0} \leq a_1 \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\sigma_0} \theta_2^{\tau_0 - \tau_0}. \tag{2.15}
\]

In particular, (2.14) and (2.15) lead to (when $\lambda = u$), respectively, \[
\left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\tau_0} \theta_1^{\tau_0 - \sigma_0} \sup_{\lambda \in E, m \geq M, n \geq N} \left[ \prod_{j=1}^{\tau_0 - \tau_0} \prod_{i=1}^{\tau_0} (c - uQ_{m-i,n-i-j}) \right] \frac{1}{\tau_0 - \tau_0} \geq \frac{u}{a_1}
\text{ for } \sigma_0 > \tau_0, \tag{2.16}
\]

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and

\[
\left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\sigma_0} \theta_2^{\sigma_0 - \tau_0} \sup_{\lambda \in E, M \geq m, N \geq n} \left[ \prod_{j=1}^{\tau_0 - \sigma_0} \frac{\sigma_0}{\tau_0} \prod_{i=1}^{\tau_0} (c - u Q_{m-i,n-i-j}) \right]^\frac{1}{\sigma_0 - \tau_0} \geq \frac{u}{a_1} \quad \text{for } \tau_0 > \sigma_0.
\]  

(2.17)

Since \( u \in S \) and \( u' \leq u \) implies that \( u' \in S \), it follows from (2.12) and (2.16) for \( \sigma_0 > \tau_0 \), (2.13) and (2.17) for \( \tau_0 > \sigma_0 \) that \( \frac{u}{a_1} \in S \). Repeating the above arguments with \( u \) replaced by \( \frac{u}{a_1 a_2} \), we get \( \frac{u}{\prod_{i=1}^{\infty} a_i} \in S \), where \( a_2 \in (0, 1) \). Continuing in this way, we obtain \( \frac{u}{\prod_{i=1}^{\infty} a_i} \in S \), where \( a_i \in (0, 1) \). This contradicts the boundedness of \( S \).

The proof is complete.

**Corollary 2.1.** In addition to (i) of Theorem 2.1, assume that for \( \sigma_0 > \tau_0 > 0 \),

\[
\lim_{m,n \to \infty} \inf \frac{1}{(\sigma_0 - \tau_0)\tau_0} \sum_{j=1}^{\tau_0 - \sigma_0} \sum_{i=1}^{\sigma_0} Q_{m-i-j,n-i-j} > \frac{c^{\sigma_0 + 1} \tau_0}{(\tau_0 + 1)^{\tau_0 + 1}} \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\tau_0} \theta_1^{\sigma_0 - \tau_0},
\]

and for \( \tau_0 > \sigma_0 > 0 \),

\[
\lim_{m,n \to \infty} \inf \frac{1}{(\tau_0 - \sigma_0)\sigma_0} \sum_{j=1}^{\tau_0 - \sigma_0} \sum_{i=1}^{\sigma_0} Q_{m-i,n-i-j} > \frac{c^{\sigma_0 + 1} \tau_0}{(\sigma_0 + 1)^{\sigma_0 + 1}} \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{-\sigma_0} \theta_2^{\tau_0 - \sigma_0}.
\]

Then every solution of (1.1) oscillates.

**Proof.** We note that

\[
\max_{\frac{\tau_0}{\tau_0} > \lambda > 0} \lambda (c - \lambda e)^{\tau_0} = \frac{c^{\tau_0 + 1} \tau_0}{e(\tau_0 + 1)^{\tau_0 + 1}}.
\]

We shall use this for

\[
e = \frac{1}{(\sigma_0 - \tau_0)\tau_0} \sum_{j=1}^{\tau_0 - \sigma_0} \sum_{i=1}^{\sigma_0} Q_{m-i-j,n-i}.
\]

Clearly,

\[
\lambda \left[ \prod_{j=1}^{\tau_0} \prod_{i=1}^{\tau_0} (c - \lambda Q_{m-i-j,n-i}) \right]^{\frac{1}{\sigma_0 - \tau_0}} \\
\leq \lambda \left[ \prod_{j=1}^{\tau_0} \sum_{i=1}^{\tau_0} (c - \lambda Q_{m-i-j,n-i}) \right]^{\frac{1}{\tau_0}} \\
\leq \lambda \left[ c - \lambda \sum_{j=1}^{\tau_0} \sum_{i=1}^{\tau_0} (Q_{m-i-j,n-i}) \right]^{\frac{1}{\sigma_0 - \tau_0}} \\
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\[
\leq \frac{c^{\tau_0+1} \tau_0}{e(\tau_0 + 1)^{\tau_0+1}} \\
< \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right) \tau_0 \theta_1^{\tau_0-\sigma_0}.
\]

Similarly, we have
\[
\lambda \left[ \prod_{j=1}^{\tau_0-\sigma_0} \prod_{i=1}^{\sigma_0} (c - \lambda Q_{m-i,n-i-j}) \right]^{1/\tau_0-\sigma_0} < \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\sigma_0} \theta_2^{\sigma_0-\tau_0}.
\]

By Theorem 2.1, every solutions of (1.1) oscillates. The proof is complete.

By a similar argument, we have the following results:

**Corollary 2.2.** If the condition of Theorem 2.2 holds, and
\[
\lim_{m,n \to \infty} \inf Q_{m,n} = q > c^{\sigma_0+1} \sigma_0^0 \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{-\sigma_0},
\]
then every solution of (1.1) oscillates.

**Theorem 2.3.** Assume that
(i) \( \lim_{m,n \to \infty} \sup Q_{m,n} > 0; \)
(ii) for \( \sigma_0, \tau_0 > 0, \)
\[
\lim_{m,n \to \infty} \inf Q_{m,n} = q > 0, \tag{2.20}
\]
and
\[
\lim_{m,n \to \infty} Q_{m,n} > c \theta_1^{\sigma_0} \theta_2^{\tau_0} - \frac{a \theta_1 + b \theta_2}{c} q > 0. \tag{2.21}
\]

Then every solution of (1.1) oscillates.

**Proof.** Suppose, to the contrary, \( A_{m,n} \) is an eventually positive solution. From (2.3) and (2.20), for any \( \epsilon > 0, \) we have \( Q_{m,n} > q - \epsilon \) for \( m \geq M, n \geq N. \) From (2.3), Lemma 2.1 and above inequality, we obtain
\[
A_{m,n} \geq \frac{(q - \epsilon)}{c} A_{m-\sigma_0,n-\tau_0} \geq \frac{(q - \epsilon)}{c} \theta_1^{1-\sigma_0} \theta_2^{1-\tau_0} A_{m-1,n-1},
\]
\[
A_{m,n} \geq \frac{(q - \epsilon)}{c} \theta_1^{1-\sigma_0} \theta_2^{1-\tau_0} A_{m-1,n}, \quad \text{and} \quad A_{m,n} \geq \frac{(q - \epsilon)}{c} \theta_1^{1-\sigma_0} \theta_2^{1-\tau_0} A_{m,n-1}.
\]

Substituting above inequalities into (2.3), we get
\[
\left[ \frac{a \theta_1^{1-\sigma_0} \theta_2^{1-\tau_0} + b \theta_1^{1-\sigma_0} \theta_2^{1-\tau_0}}{c} (q - \epsilon) - c + Q_{m,n} \theta_1^{1-\sigma_0} \theta_2^{1-\tau_0} \right] A_{m,n} < 0,
\]
which implies
\[
\lim_{m,n \to \infty} Q_{m,n} \leq c \theta_1^{\sigma_0} \theta_2^{\tau_0} - \frac{a \theta_1 + b \theta_2}{c} q > 0.
\]

This contradicts (2.21). The proof is complete.
Theorem 2.4. Assume that
(i) \( \lim_{m,n \to \infty} \sup Q_{m,n} > 0 \);
(ii) \( \sigma_0 = \tau_0 = 0 \), and
\[
\lim_{m,n \to \infty} \sup Q_{m,n} > c. \tag{2.22}
\]
Then every solution of (1.1) oscillates.

Proof. Let \( u \in S \). Then from (2.3) and Lemma 2.1, we have \( -c + Q_{m,n}A_{m,n} < 0 \), which implies \( \lim_{m,n \to \infty} \sup Q_{m,n} \leq c \). This contradicts (2.22). The proof is complete.

REFERENCES