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A Class of '*n*' Distant Graceful Trees

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Abstract

In this paper we show that a tree T with the following properties have graceful labeling.
1] T has a path H such that every pendant vertex of T has distance n (a fixed positive integer) from H.
2] Every vertex of T excluding one end vertex of H has even degree.

Key words

Graceful labeling; n distant tree; Component moving transformation; Transfer of the first type

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1. INTRODUCTION

Definition 1.1 A *graceful labeling* of a tree T with q edges is a bijection $f : V(T) \to \{0, 1, 2, \ldots\ldots q\}$ such that $\{|f(u) - f(v)| : \{u, v\} \text{ is an edge of } T\} = \{1, 2, 3, \ldots\ldots, q\}$. A tree which has a graceful labeling is called a *graceful tree*.

Definition 1.2 A Tree T is said to be a *n distant tree* if it possesses at least two vertices at '$2n + 1$' distance apart. We observe that a n distant tree has the path $H = x_0, x_1, \ldots\ldots x_m$ such that for $i = 1, 2, 3, \ldots\ldots, m-1$, the distance of each vertex in $T - \{x_1, \ldots\ldots, x_{i-1}, x_{i+1}, \ldots\ldots, x_{m-1}\}$ from x_i is at most n, the distance of at least one vertex in $T - \{x_1, x_2, \ldots\ldots\ldots\ldots, x_{m-1}, x_m\}$ from x_0 is n, and the distance of at least one vertex in $T - \{x_0, x_1, \ldots\ldots, x_{m-2}, x_{m-1}\}$ from x_m is n. We call the path $H = x_0 x_1 \ldots\ldots x_m$ as the central path of n distance tree T. The tree in Figure 1 is a four distance tree with each of its pendant vertices is at distance 4 from the central path.

In a bid to resolve the conjecture of Ringel and Kotzig[5] involving graph decomposition that the complete graph K_{2n+1} decomposes into $2n + 1$ copies of a tree with n edges, the concept of β − *valuation* emerged.

In 1964 Ringel and Kotzig[5] conjectured that every tree has β valuation. The above conjecture is popularly known as 'Graceful Tree Conjecture'. In 1966 Rosa[6] proved that the complete graph K_{2n+1} decomposes into $2n+1$ copies of a tree with n edges provided that the tree has a β valuation. As a consequence of Rosa's findings many researchers and workers of graph theory got inspired to work more exhaustively on the graceful tree conjecture. In 1972 Golomb[2] called β valuation as graceful labeling, which is now the popular term. Though the conjecture is unresolved till date, there have been numerous efforts to resolve the graceful tree conjecture for last five decades. One can refer to Gallian's latest survey on graph labeling problems[1] on the progress made in resolving the graceful tree conjecture. Here we are inspired by the transformation techniques discussed by Herncier and Havier[3] in which they proved all trees up to diameter five are graceful. In this paper we give graceful labeling to a class of n distant trees using the transformation techniques presented in[3]. The n distant trees with central path $H = x_0x_1x_2 \ldots \ldots x_m$ to which we give graceful labeling in this paper have the characteristics that each non pendant vertex of the tree excluding x_m has even degree and each pendant vertex is at a distance n from the central path H. Figure 1 is a (graceful) 4-distance tree satisfying the conditions mentioned above.

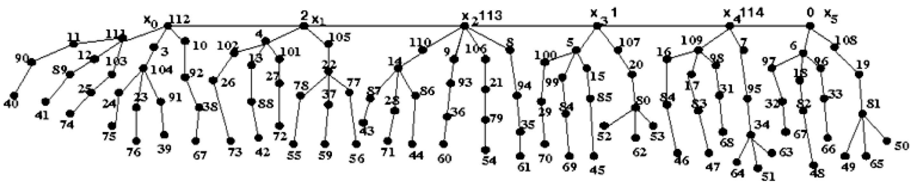

Figure 1

A Four Distance Tree in Which Each Non-Pendant Vertex has Even Degree Except the Vertex with Label 0

In order to prove our main result we need some existing terminologies and results as described below.

Definition: 1.3 For an edge $e = \{u, v\}$ of a tree T, we define $u(T)$ as that connected component of $T - e$ which contains the vertex u. Here we say $u(T)$ is a component incident on the vertex v. If a and b are vertices of a tree T, $u(T)$ is a *component incident on* a and $b \notin u(T)$ then deleting the edge $\{a, u\}$ from T and making b and u adjacent is called the *component moving transformation.* Here we say the component $u(T)$ has been transferred or moved from a to b.

Definition: 1.4 Let T be a labelled tree and a and b be two vertices of T, and a be attached to some components. The $a \to b$ transfer is called *a transfer of the first type* if the labels of the transferred components constitute a set of consecutive integers.

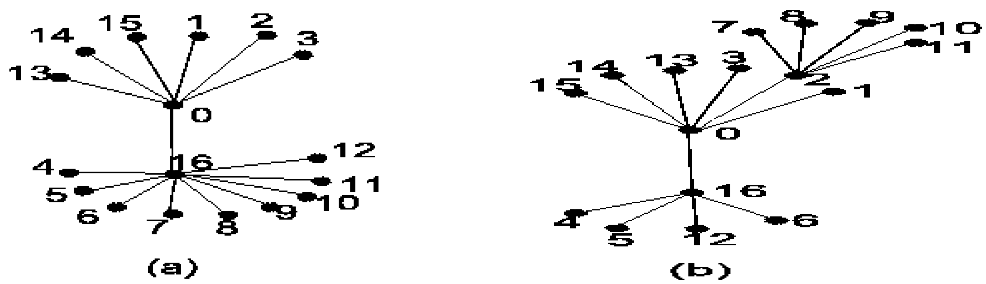

Figure 2

The Tree in (a) is a Tree with a Graceful Labeling. The Tree in (b) is Obtained from (a) by Applying a Transfer of the First Type $16 \to 2$

Notation 1.5 For any two vertices a and b of a tree T, the notation $a \to b$ *transfer means* that we move some components incident on the vertex a to the vertex b. If we consider successive transfers $a \to b, b \to c, c \to d, \ldots \ldots$, we simply write $a \to b \to c \to d \ldots \ldots$ transfer. In a transfer $a_1 \to a_2 \to a_3 \ldots \ldots \to a_n$, we call each vertex except a_n a *vertex of the transfer*.

Lemma 1.6 [4] Let f be a graceful labeling of a tree T; let a and b be two vertices of T; let $u(T)$ and $v(T)$ be two components incident on a, where $b \notin u(T) \cup v(T)$. Then the following hold:

(i) If $f(u) + f(v) = f(a) + f(b)$ then the tree T^*; obtained from T by moving the components $u(T)$ and $v(T)$ from a to b is also graceful.

(ii) If $2f(u) = f(a) + f(b)$ then the tree T^{**} obtained from T by moving the component $u(T)$ from a to b is also graceful.

2. RESULTS

Lemma 2.1 In a graceful labeling f of a graceful tree T, let a and b be the labels of two vertices. Let a be attached to a set of vertices whose labels are consecutive integers constituting a sequence $S = (n, n+p, n+1, n+p-1, n+2, n+p-2, \ldots\ldots)$ with $\max\{S\} = n+p$, $\min\{S\} = n$, and the sums of the consecutive terms of S are alternately $a+b$ and $a+b-1$ beginning with the sum $a+b-1$. By making a transfer $a \to b$ of the first type we can keep any odd number of terms from the beginning of S at a and move the rest to b, and the resultant tree thus formed will be graceful.

Proof: The terms of S are consecutive integers $n, n+1, n+2, \ldots\ldots n+p$. We observe that on removing any number of terms from the beginning of S, the remaining terms form a set of consecutive integers. Moreover, we have $(n+i)+(n+p-i+1) = a+b, i = 1, 2, \ldots\ldots, i.e$ excluding n the terms of S can be paired whose sum is $a+b$. Now we carry out a transfer $a \to b$ of the first type keeping any odd number of terms from the beginning, say the first $2r+1$ terms of S, namely $n, n+i, n+p-i+1, i = 1, 2, 3, \ldots\ldots, r$, at a, and transfer the rest to b. Let A_1 be the set of terms of S which have been moved to b. Now the elements of A_1 are consecutive integers with the property that for each $z \in A_1$, either we have $2z = a+b$ or there is another (unique) element w in A_1 such that $z+w = a+b$. The new tree thus formed is graceful by Lemma 1.6.

Lemma 2.2 In a graceful labeling f of a graceful tree T, let a and b be the labels of two vertices. Let a be attached to a set of vertices whose labels are consecutive integers constituting a sequence $S = (n+p, n, n+p-1, n+1, n+p-2, n+2, \ldots\ldots)$ with $\max\{S\} = n+p$, $\min\{S\} = n$, and the sums of the consecutive terms of S are alternately $a+b$ and $a+b+1$ beginning with the sum $a+b+1$. By making a transfer $a \to b$ of the first type we can keep any odd number of terms from the beginning of S at a and move the rest to b, and the resultant tree thus formed will be graceful.

Proof: The terms of S are consecutive integers $n, n+1, n+2, \ldots\ldots n+p$. We observe that on removing any number of terms from the beginning of S, the remaining terms form a set of consecutive integers. Moreover, we have $(n+p-i)+(n+i-1) = a+b, i = 1, 2, \ldots\ldots, i.e$ excluding $n+p$ the terms of S can be paired whose sum is $a+b$. Now we carry out a transfer $a \to b$ of the first type keeping any odd number of terms from the beginning, say the first $2r+1$ terms of S, namely $n+p, n+i-1, n+p-i, i = 1, 2, 3, \ldots\ldots, r$, at a, and transfer the rest to b. Let A_2 be the set of terms of S which have been moved to b. Now the elements of A_2 are consecutive integers with the property that for each $z \in A_2$, either we have $2z = a+b$ or there is another (unique) element w in A_2 such that $z+w = a+b$. The new tree thus formed is graceful by Lemma 1.6.

Observation 2.3(a) Consider any pair of vertex labels a and b in a graceful tree, where a is attached to a set of vertices with labels as in Lemma 2.1. After we carry out a transfer a to b of the first type as in Lemma 2.1, the set A_1 of the vertex labels of S that are transferred to b is of the form $A_1 = \{n+r+1, n+r+2, \ldots\ldots, n+r_1\}$, $r_1 = p-r$ with $(n+r+1+i)+(n+r_1-i) = a+b, i = 0, 1, 2, \ldots\ldots$. Further, by repairing the elements of A_1, we get, $(n+r+1+i)+(n+r_1-i) = a+b-1$, for $0 \le i \le [\frac{r_1-r+1}{2}]$. That is the elements of A_1 form the sequence $(n+r_1, n+r+1, n+r_1-1, n+r+2, \ldots)$. Therefore, next if we make a transfer $b \to a-1$ of the first type, then the set A_1 and the vertices (or labels) b and $a-1$ satisfy the hypothesis of Lemma 2.2.

(b) Consider any pair of vertex labels a and b in a graceful tree, where a is attached to a set of vertices whose labels are as in Lemma 2.2. After we carry out a transfer $a \to b$ of the first type as in lemma 2.2, the set A_2 of the vertex labels of S that are transferred to b is of the form $A_2 = \{n+r, n+r+1, \ldots\ldots, n+r_2\}, r_2 = p-r-1$ with $(n+r+i)+(n+r_2-i) = a+b$, where $0 \le i \le [\frac{r_2-r+1}{2}]$.

Further, by re-pairing the elements of A_2, we get, $(n+r+1+i)+(n+r_2-i) = a+b+1$, for $0 \le i \le [\frac{r_2-r+1}{2}]$. That is the elements of A_2 form the sequence $(n+r_2, n+r, n+r_2-1, n+r+1, n+r_2-2, , n+r+2, \ldots)$. Therefore, next if we make a transfer $b \to a+1$, then the set A_2 and the vertices b and $a+1$ satisfy the hypothesis of Lemma 2.2.

(c) In this paper we carry out sequence of (successive) transfers of the first type either of form $a \to b \to a-1 \to b+1 \to a-2 \to b+2 \to \cdots \to z, z = a-p_1$ or $b+p_2$ or of the form $a \to b \to a+1 \to b-1 \to a+2 \to b-2 \to \cdots w, w = a-r_1$ or $b+r_2$ and accordingly the sequence of vertex labels incident on a are consecutive integers as in Lemma 2.1 or Lemma 2.2. In view of our observations (a) and (b) we carry out the sequence (successive) transfer of the first type $a \to b \to a-1 \to b+1 \to a-2 \to b+2 \to \cdots z, z = a-p_1$

or $b + p_2$(or $a \to b \to a+1 \to b-1 \to a+2 \to b-2 \to \cdots w, w = a - r_1$ or $b + r_2$) by applying Lemma 2.1 and Lemma 2.2 alternately beginning with Lemma 2.1(or Lemma 2.2)

By applying Lemma 2.1 and Lemma 2.2 alternately beginning with Lemma2.1(respectively, Lemma 2.2) as discussed in observation 2.3, we get the following two results.

Lemma 2.4 In a graceful labeling f of a tree T, let $a, a-1, a-2, \ldots, a-p_1, b, b+1, b+2, \ldots\ldots, b+p_2$ be some vertex labels. Let a be attached to a set of vertices (or components) whose labels are consecutive integers consisting a sequence $S = \{n, n+p, n+1, n+p-1, n+2, n+p-2, \ldots\ldots\}$ with $\max\{S\} = n+p$, $\min\{S\} = n$, and the sums of the consecutive terms of S are alternately $a+b$ and $a+b-1$ begining with the sum $a+b-1$. By making a sequence of transfers of the first type $a \to b \to a-1 \to b+1 \to a-2 \to b+2 \to \ldots \to z$, where $z = a - p_1$ or $b + p_2$, an odd number of terms from the beginning of S are kept at a, the next odd number of terms of S are kept at b, the next odd number of terms of S are kept at $a-1$, the next odd number of terms of S are kept at $b+1$ and so on. The resultant tree thus formed will be graceful.

Lemma 2.5 In a graceful labeling f of a tree T, let $a, a+1, a+2, \ldots\ldots, a+r_1, b, b-1, b-2, \ldots\ldots, b-r_2$ be some vertex labels. Let a be attached to a set of vertices (or components) whose labels are consecutive integers consisting the sequence $S = \{n+p, n, n+p-1, n+1, n+p-2, n+2, \ldots\ldots\}$ with $\max\{S\} = n+p$, $\min\{S\} = n$, and the sums of the consecutive terms of S are alternately $a+b$ and $a+b+1$ begining with the sum $a+b+1$. By making a sequence of transfers of the first type $a \to b \to a+1 \to b-1 \to a+2 \to b-2 \to \cdots \to w$, where $w = a + r_1$ or $b - r_2$, an odd number of terms from the beginning of S are kept at a, the next odd number of terms of S are kept at b, the next odd number of terms of S are kept at $a+1$, the next odd number of terms of S are kept at $b-1$ and so on. The resultant tree thus formed will be graceful.

Next we give the most significant result of this article. Here we give graceful labeling to a class of $n-$distant tree with degree of each non-pendant vertex excluding one end of the central path has even degree.

Theorem 2.6 A n-distant tree with the central path $H = x_0, x_1, \ldots. x_m$ with the following properties admits graceful labeling.

i) Each non pendant vertex of the tree excluding x_m has even degree.

ii) Each pendant vertex of the tree lies at a distance n from the central path H.

Proof: Let T be a n distant tree with $|E(T)| = q$ and the central path H satisfy the properties (i) and (ii) mentioned above. Let the number of neighbours of x_0 in $T - H$ be $2k_0 + 1$ and for $i = 1, 2, \ldots, m$, the number of neighbours of x_i in $T - H$ be $2k_i$. Let $k^{(1)} = k_0 + k_1 + k_2 + \ldots + k_m$. The number of vertices of T at distance one from H is $2k^{(1)} + 1$, *i.e.* an odd number. Since each non pendant vertex of T at a distance $j, 1 \le j \le n-1$ from H has an even degree, it is attached to exactly one vertex which at a distance $j-1$ from H and an odd number of vertices which are at a distance $j+1$ from H. Since number of vertices of T at distance one from H is odd, and each non pendant vertex at distance $j, 1 \le j \le n-1$ is attached to an odd number of vertices at distance $j+1$, the number of vertices of T at distance $j, j \ge 2$ from H is an odd number, say $2k^{(j)} + 1$. Here we have $q = m + \sum_{j=1}^{n}(2k^{(j)} + 1)$. We proceed as per the following steps to give graceful labeling to T.

Step:1 We first form the graceful tree $G(T)$ as shown in Figure 3 and with $|E(G(L))| = q + 1$, i.e we attach a new pendant vertex $x_m + 1$ to the vertex x_m, the degree of each vertex x_i, $1 \le i \le m$, is two, and x_0 is attached to $q - m$ pendant vertices. We consider the following graceful labeling of $G(T)$.

If m is even:

$$f(v) = \begin{cases} \dfrac{m}{2} - i, v = x_{2i}, i = 0, 1, 2, \ldots, \dfrac{m}{2} \\ q - \dfrac{m}{2} + 1 + i, v = x_{2i+1}, i = 0, 1, 2, \ldots, \dfrac{m}{2} \\ r, r = \dfrac{m}{2} + 1, \dfrac{m}{2} + 2, \ldots, q - \dfrac{m}{2} \\ \text{for the } q - m \text{ pendant vertices adjacent to } x_0. \end{cases} \tag{2.1}$$

If m is odd:

$$f(v)=\begin{cases}\dfrac{m-1}{2}-i, v=x_{2i+1}, i=0,1,2,\ldots,\dfrac{m-1}{2}\\ q-\dfrac{m-1}{2}+i, v=x_{2i}, i=0,1,2,\ldots,\dfrac{m+1}{2}\\ r, r=\dfrac{m-1}{2}+1,\dfrac{m-1}{2}+2,\ldots,q-\dfrac{m-1}{2}-1\\ \text{for the } q-m \text{ pendant vertices adjacent to } \mathrm{x}_0.\end{cases} \tag{2.2}$$

Let A_0 be the set of all pendant vertices adjacent to x_0 in $G(T)$. The set A_0 can be written as $A_0=\{a_1,a_2,\ldots,a_{q-m}\}$, where, for $1\le s\le q-m$, $a_s=\begin{cases} q-\frac{m}{2}+1-s\,if\ m\ is\ even\\ \frac{m-1}{2}+s\ if\ m\ is\ odd\end{cases}$.

Further, the elements of A_0 are consecutive integers satisfying

$$a_s+a_{q-m+1-s}=f(x_0)+f(x_1)=\begin{cases} q+1\ if\ m\ is\ even\\ q\ if\ m\ is\ odd\end{cases}\quad \text{for } 1\le s\le\left[\frac{q-m+1}{2}\right].$$

Step:2 We keep k_0 pairs $(a_s,a_q-m+1-s)$, $s=1,2,\ldots k_0$, at x_0 and move the rest to x_1 and let the tree thus formed be G_1. The tree G_1 has the graceful labeling f by Lemma 1.6. Let A_1 denote the set of vertices in A_0 that are transferred to x_1, i.e. $A_1=\{a_{k_0+1},a_{k_0+2},\ldots,a_{q-m-k_0}\}$.

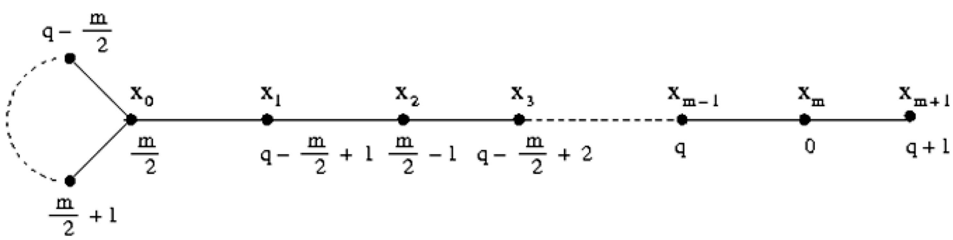

Figure 3
The tree $G(L)$ corresponding to the lobster L for the case m is even

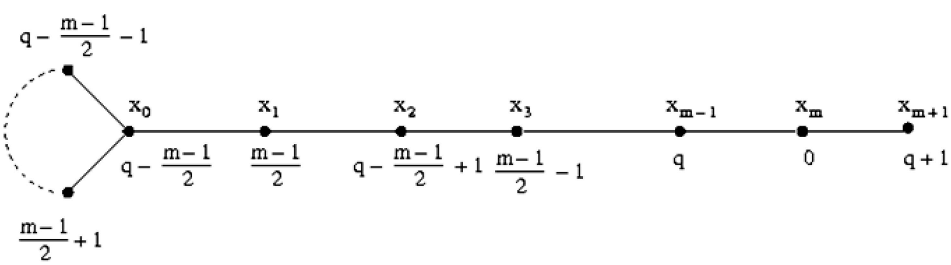

Figure 4
The tree $G(L)$ corresponding to the lobster L for the case m is odd

Step:3 Consider the sequence of transfers of the first type $x_1\to x_2\ldots\to x_m\to x_{m+1}$. The elements of A_1 are consecutive integers satisfying

$$a_s+a_{q-m+1-s}=f(x_0)+f(x_1)=\begin{cases} q+1\ if\ m\ is\ even\\ q\ if\ m\ is\ odd\end{cases}\quad for\ k_0+1\le s\le\left[\frac{q-m-k_0+1}{2}\right],$$

i.e the elements of A_1 form the sequence $S_1=(a_{k_0+1},a_{q-m-k_0},a_{k_0+2},a_{q-m-k_0-1},\ldots)$ whose sums of consecutive terms are alternately q and $q+1$ beginning with the sum $q+1$ if m is even and q if m is odd .We observe that the labels of the vertices $x_i, 1\le i\le m+1$ of the transfer and the terms of the sequence S_1 satisfy the properties of the vertices of transfer and the sequence S of Lemma 2.5 if m is even and Lemma 2.4 if m is odd. We carry out the transfer $x_1\to x_2\to\ldots\ldots\to x_m\to x_{m+1}$ keeping $2k_1-1$ terms from the beginning of S_1 at x_1, the next $2k_2 - 1$ terms from S_1 at x_2, the next $2k_3 - 1$ terms from S_1 at x_3, and so on. The resultant tree, say G_2 thus formed has a graceful labeling f either by Lemma 2.5 (if m is even) or Lemma 2.4 (if m is odd).

Step:4 Let S_2 be the sequence of terms of S_1 that have come to the vertex x_{m+1} after the previous step. Obviously, the terms of S_2 are consecutive integers, Moreover, we observe that

$$S_2 = \begin{cases} (a_{k^{(1)}-\frac{m}{2}+1}, a_{q-k^{(1)}-\frac{m}{2}}, a_{k^{(1)}-\frac{m}{2}+2}, a_{q-k^{(1)}-\frac{m}{2}-1}, a_{k^{(1)}-\frac{m}{2}+3}, a_{q-k^{(1)}-\frac{m}{2}-2}, \cdots) & if\ m\ is\ even \\ (a_{q-k^{(1)}-\frac{m-1}{2}}, a_{k^{(1)}-\frac{m+1}{2}+2}, a_{q-k^{(1)}-\frac{m-1}{2}-1}, a_{k^{(1)}-\frac{m+1}{2}+3}, a_{q-k^{(1)}-\frac{m-1}{2}-2}, a_{k^{(1)}-\frac{m+1}{2}+4}, \cdots) & if\ m\ is\ odd \end{cases}$$

We carry out a transfer $x_m + 1 \to x_m$, i.e. $q + 1 \to 0$, of the first type and bring back all the terms of S_2 to x_m. Then we remove the vertex $x_m + 1$. Obviously, the new tree thus formed, say G_3, is graceful.

Step:5 Next consider the transfer $x_m \to x_{m-1} \to x_m - 2 \to \cdots x_1 \to x_0 \to a_1$. We notice that sums of consecutive terms of S_2 are q and $q - 1$ beginning with the sum q. So the sequence S_2 and the transfer $x_m \to x_{m-1} \to x_{m-2} \to \cdots \to x_1 \to x_0 \to a_1$ satisfy the properties of the sequence S and the transfer $a \to b \to a+1 \to b-1 \to a+2 \to b-2 \to \cdots \to w$, where $w = a - r_1$ or $b + r_2$ in Lemma 2.5. Using Lemma 2.5 we carry out the transfer $x_m \to x_{m-1} \to x_{m-2} \to \cdots x_1 \to x_0 \to a_1$ of the first type keeping exactly one term of S_2 at each vertex of transfer.

Step:6 Let $t = \sum_{j=1}^{n-1}(2k^{(j)} + 1)$. Consider the transfer $a_1 \to a_{q-m} \to a_2 \to a_{q-m-1} \to \cdots \to a_p$, where $p = q-m-\frac{t}{2}+1$ if t is even and $\frac{t-1}{2}+1$ if t is odd. Let S_3 be the sequence of terms of S_2 which have come to the vertex a_1 after the transfer in step-5. We observe that $S_3 = (a_{q-m-k^{(1)}}, a_{k^{(1)}+2}, a_{q-m-k^{(1)}-1}, a_{k^{(1)}+3}, q_{q-m-m^{(1)}-2}, a_{k^{(1)}+4}, \cdots)$. The terms of S_3 are consecutive integers whose sums are alternatively q and $q + 1$ beginning with the sum q if m is even and $q + 1$ if m is odd. The sequence S_3 and the transfer $a_1 \to a_{q-m} \to a_2 \to a_{q-m-1} \to \cdots \to a_p$ resemble the sequence S and the transfer $a \to b \to a-1 \to b+1 \to a-2 \to b+2 \to \cdots z$ of Lemma 2.4 if m is even and the sequence S and the transfer $a \to b \to a+1 \to b-1 \to a+2 \to b-2 \to \cdots \to w$ of Lemma 2.5 if m is odd. By Lemma 2.4 or Lemma 2.5 we can give a graceful labeling to T by carrying out the transfer $a_1 \to a_{q-m} \to a_2 \to a_{q-m-1} \to \cdots \to a_p$ of the vertices whose labels are the terms of S_3 keeping desired odd number of terms at each vertex of the transfer. Hence the proof.

Example: Figure 1 is a graceful 4-distant tree satisfying the properties (i) and (ii) of Theorem 2.6. The graceful labeling of the tree is obtained if we proceed as per steps 1 to 6 above of the proof of Theorem 2.6. The central Path is $= x_0x_1x_2x_3x_4x_5$, *i.e.* $m = 5$. Here $q = 114, k_0 = 1, k_1 = 1, k_2 = 2, k_3 = k_4 = k_5 = 1$; $k^{(1)} = 7$, $k^{(2)} = 13, k^{(3)} = 15, k^{(4)=18}$. Here for $s = 1, 2, \ldots, 109(= q - m), a_s = \frac{m-1}{2} + s = 2 + S$, $A_0 = \{a_1, a_2, \ldots, a_{109}\} = \{3, 4, \ldots, 110, 111\}$, $A_1 = \{a_{k_0+1}, a_{k_0+2}, \ldots, a_{q-m-k_0}\} = \{a_2, a_3, \ldots, a_108\} = \{4, 5, \ldots 110\}$,

$$S_1 = (a_{k_0+1}, a_{q-m-k_0}, a_{k_0+2}, a_{q-m-k_0-1}, \ldots) = (a_2, a_{108}, a_3, a_{107}, \ldots) = (4, 110, 5, 109, \ldots)$$
$$S_2 = (a_{q-k^{(1)}-\frac{m-1}{2}}, a_{k^{(1)}-\frac{m+1}{2}+2}, a_{q-k^{(1)}-\frac{m-1}{2}-1}, a_{k^{(1)}-\frac{m+1}{2}+3}, a_{q-k^{(1)}-\frac{m-1}{2}-2}, a_{k^{(1)}-\frac{m+1}{2}+4}, \cdots)$$
$$= (a_{105}, a_6, a_{104}, a_7, a_{103}, a_8, \ldots) = (107, 8, 106, 9, 105, 10, \cdots)$$
$$S_3 = (a_{q-m-m^{(1)}}, a_{k^{(1)}+2}, a_{q-m-k^{(1)}-1}, a_{k^{(1)}+3}, a_{q-m-k^{(1)}+4}, \ldots)$$
$$= (a_{102}, a_9, a_{101}, a_10, a_{100}, a_{11}) = (104, 11, 103, 12, 102, 14, \ldots)$$
$$t = \sum_{j=1}^{n-1}(2k^{(j)} + 1) = 73.$$

So $p = \frac{t-1}{2} + 1 = 37$. The transfer $x_1 \to x_2 \to \cdots \to x_m \to x_{m+1}$ in step 3 is the transfer $3 \to 112 \to 2 \to 113 \to 1 \to 114 \to 0 \to 115$, the transfer $x_m \to x_{m-1} \to \cdots \to x_1 \to x_0 \to a_1$ in step 5 is the transfer $0 \to 114 \to 1 \to 113 \to 2 \to 112 \to 3$; and the transfer $a_1 \to a_{q-m} \to a_2 \to a_{q-m-1} \to \cdots \to a_p$ in step 6 is the transfer $a_1 \to a_{109} \to a_2 \to a_{108} \to \cdots \to a_{37}$, i.e. the transfer $3 \to 111 \to 4 \to 110 \to \cdots \to 39$.

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