Asymptotic Solutions of Fifth Order Over-Damped Nonlinear Systems with Cubic Nonlinearity

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Abstract: Many engineering problems and physical systems of fifth degrees of freedom are governed by the fifth order nonlinear differential equations which are over-damped. In this article a fifth order nonlinear differential equation modeling an over-damped symmetrical system is considered. A perturbation technique based on the KBM method and the work of Akbar et al. is developed for obtaining the transient response when the eigenvalues are in integral multiple. The results obtained by the presented technique agree with those results obtained by the numerical method nicely. An example is solved to illustrated method.

Keywords: Nonlinearity; Over-damped systems; Perturbation; Eigenvalues

INTRODUCTION

The asymptotic method of Krylov–Bogoliubov–Mitropolskii (KBM)⁶,⁸ is particularly convenient, and one of the widely-used tools to obtain solutions of nonlinear systems with small nonlinearities. The method originally developed by Krylov and Bogoliubov⁶, for systems with periodic solutions with small nonlinearities, was later amplified and justified by Bogoliubov and Mitropolskii⁶. Popov¹⁵ extended the method to nonlinear systems affected by strong linear damping forces. Owing to physical importance Popov’s results were rediscovered by Mendelson⁹. Later, Murty and Deekshatulu¹¹ extended the method to over-damped nonlinear systems. Sattar¹⁶ has studied second order critically-damped nonlinear systems by making use of the KBM method. Murty¹³ presented a unified KBM method for second order nonlinear systems which covers the undamped, the over-damped and the damped oscillatory cases. First, Osiniskii¹⁴ extended the KBM method to solve third-order nonlinear differential systems using some restrictions, which make the solution over-simplified. Mulholland¹⁰ had removed these restrictions and found desired solutions of third-order nonlinear systems. Sattar¹⁷ investigated solutions of three-dimensional over-damped nonlinear systems. Shamsul¹⁸ presented an asymptotic method for second-order over-damped and critically-damped nonlinear systems. Then Shamsul¹⁰ extended the method presented in¹⁸ to third-order over-damped nonlinear systems under some special conditions.

In article¹² Murty et al. also extended the KBM method for solving fourth-order over-damped nonlinear
systems which was a simple (particular) problem. Akbar et al.\textsuperscript{[1]} generalized the method presented in\textsuperscript{[12]}. Akbar et al.\textsuperscript{[1]} also show that their method is easier than the method of Murty et al.\textsuperscript{[12]}. Then Akbar et al.\textsuperscript{[2]} extended the method presented in\textsuperscript{[1]} to damped oscillatory nonlinear systems. Akbar et al.\textsuperscript{[3]} also presented a simple technique for solving fourth order over-damped nonlinear systems. Akbar and Uddin\textsuperscript{[5]} found solutions of an over-damped system in the case of special damping forces. Habibur et al.\textsuperscript{[7]} investigated solutions of certain forth order damped oscillatory systems.

Very recently, Akbar and Tanzer\textsuperscript{[6]} extended the KBM method for solving the fifth order oscillatory nonlinear systems with small nonlinearities.

In this article, an asymptotic solution of fifth-order over-damped symmetrical nonlinear system is investigated, based on the KBM method and the work of Akbar et al.\textsuperscript{[3]}. The results obtained by the presented technique show good coincidence with numerical results obtained by the fourth-order Runge-Kutta method.

1. THE METHOD

Consider a nonlinear symmetrical over-damped system governed by the fifth order differential equation:

\[
d^5x dt^5 + k_1 \frac{d^4x}{dt^4} + k_2 \frac{d^3x}{dt^3} + k_3 \frac{d^2x}{dt^2} + k_4 \frac{dx}{dt} + k_5 x = -\varepsilon f(x)
\]  

(1)

where \(k_1, k_2, k_3, k_4, k_5\) are characteristics parameter, \(f\) is such a nonlinear function that the system (1) becomes symmetrical and \(\varepsilon\) is a small parameter. When \(\varepsilon = 0\), the equation (1) becomes linear, let us consider the five real and negative eigenvalues of the linear equation are \(-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4, -\lambda_5\). Here, the over-damping forces are represented by the real and negative eigenvalues. In this case, the solution of the linear equation is:

\[
x(t, 0) = a_{1,0} e^{-\lambda_1 t} + a_{2,0} e^{-\lambda_2 t} + a_{3,0} e^{-\lambda_3 t} + a_{4,0} e^{-\lambda_4 t} + a_{5,0} e^{-\lambda_5 t}
\]

(2)

where \(a_{1,0}, a_{2,0}, a_{3,0}, a_{4,0}\) and \(a_{5,0}\) are arbitrary constants.

When \(\varepsilon \neq 0\), following Murty and Deekshatulu\textsuperscript{[11]} and Shamsul\textsuperscript{[19]}, we seek a solution of equation (1) in an asymptotic expansion of the form:

\[
x(t, \varepsilon) = a_{1,1} e^{-\lambda_1 t} + a_{2,1} e^{-\lambda_2 t} + a_{3,1} e^{-\lambda_3 t} + a_{4,1} e^{-\lambda_4 t} + a_{5,1} e^{-\lambda_5 t} + \varepsilon u_1 (a_{1,2}, a_{2,2}, a_{3,2}, a_{4,2}, a_{5,2}, t) + \varepsilon^2 \ldots
\]

(3)

where \(a_j, j = 1, 2, \ldots, 5\) satisfy the first-order differential equations:

\[
\dot{a}_j = \varepsilon A_1 (a_{1,2}, a_{2,2}, a_{3,2}, a_{4,2}, a_{5,2}, t) + \varepsilon^2 \ldots
\]

(4)

Differentiating (3) five times with respect to \(t\), substituting (3) and the derivatives in the original equation (1), using relations of (4), and finally extracting the coefficients of \(\varepsilon\), we obtain:

\[
\begin{align*}
&+ e^{-\lambda_1 t} \left( \frac{d}{dt} + \lambda_2 - \lambda_1 \right) \left( \frac{d}{dt} + \lambda_3 - \lambda_1 \right) \left( \frac{d}{dt} + \lambda_4 - \lambda_1 \right) \left( \frac{d}{dt} + \lambda_5 - \lambda_1 \right) A_1 \\
&+ e^{-\lambda_2 t} \left( \frac{d}{dt} + \lambda_1 - \lambda_2 \right) \left( \frac{d}{dt} + \lambda_3 - \lambda_2 \right) \left( \frac{d}{dt} + \lambda_4 - \lambda_2 \right) \left( \frac{d}{dt} + \lambda_5 - \lambda_2 \right) A_2 \\
&+ e^{-\lambda_3 t} \left( \frac{d}{dt} + \lambda_1 - \lambda_3 \right) \left( \frac{d}{dt} + \lambda_2 - \lambda_3 \right) \left( \frac{d}{dt} + \lambda_4 - \lambda_3 \right) \left( \frac{d}{dt} + \lambda_5 - \lambda_3 \right) A_3 \\
&+ e^{-\lambda_4 t} \left( \frac{d}{dt} + \lambda_1 - \lambda_4 \right) \left( \frac{d}{dt} + \lambda_2 - \lambda_4 \right) \left( \frac{d}{dt} + \lambda_3 - \lambda_4 \right) \left( \frac{d}{dt} + \lambda_5 - \lambda_4 \right) A_4 \\
&+ e^{-\lambda_5 t} \left( \frac{d}{dt} + \lambda_1 - \lambda_5 \right) \left( \frac{d}{dt} + \lambda_2 - \lambda_5 \right) \left( \frac{d}{dt} + \lambda_3 - \lambda_5 \right) \left( \frac{d}{dt} + \lambda_4 - \lambda_5 \right) A_5 \\
&+ \left( \frac{d}{dt} + \lambda_1 \right) \left( \frac{d}{dt} + \lambda_2 \right) \left( \frac{d}{dt} + \lambda_3 \right) \left( \frac{d}{dt} + \lambda_4 \right) \left( \frac{d}{dt} + \lambda_5 \right) u_1 = -f^{(0)} (a_{1,2}, a_{2,2}, a_{3,2}, a_{4,2}, a_{5,2}, t)
\end{align*}
\]

(5)
where \( f^{(0)} = f(x_0) \) and \( x_0 = \sum_{j=1}^{5} a_j(t) e^{-\lambda_j t} \).

In general, the functional \( f^{(0)} \) can be expanded in the Taylor series (see also \[^1\][^2\] for details) as:

\[
f^{(0)} = \sum_{i_1=0}^{\infty} \cdots \sum_{i_5=0}^{\infty} F_{i_1} \cdots i_5(a_1, a_2, a_3, a_4, a_5) e^{-(i_1 A_1 + \cdots + i_5 A_5) t} \tag{6}
\]

Substituting the value of \( f^{(0)} \) from (6) into (5), we obtain:

\[
e^{-A_1 t} \left( \frac{d}{dt} + A_2 - A_1 \right) \left( \frac{d}{dt} + A_3 - A_1 \right) \left( \frac{d}{dt} + A_4 - A_1 \right) \left( \frac{d}{dt} + A_5 - A_1 \right) u_1
\]

\[
+ e^{-A_1 t} \left( \frac{d}{dt} + A_1 - A_2 \right) \left( \frac{d}{dt} + A_3 - A_2 \right) \left( \frac{d}{dt} + A_4 - A_2 \right) \left( \frac{d}{dt} + A_5 - A_2 \right) u_2
\]

\[
+ e^{-A_1 t} \left( \frac{d}{dt} + A_1 - A_3 \right) \left( \frac{d}{dt} + A_2 - A_3 \right) \left( \frac{d}{dt} + A_4 - A_3 \right) \left( \frac{d}{dt} + A_5 - A_3 \right) u_3
\]

\[
+ e^{-A_1 t} \left( \frac{d}{dt} + A_1 - A_4 \right) \left( \frac{d}{dt} + A_2 - A_4 \right) \left( \frac{d}{dt} + A_3 - A_4 \right) \left( \frac{d}{dt} + A_5 - A_4 \right) u_4
\]

\[
+ e^{-A_1 t} \left( \frac{d}{dt} + A_1 - A_5 \right) \left( \frac{d}{dt} + A_2 - A_5 \right) \left( \frac{d}{dt} + A_3 - A_5 \right) \left( \frac{d}{dt} + A_4 - A_5 \right) u_5
\]

\[
= - \sum_{i_1=0}^{\infty} \cdots \sum_{i_5=0}^{\infty} F_{i_1} \cdots i_5(a_1, a_2, a_3, a_4, a_5) e^{-(i_1 A_1 + \cdots + i_5 A_5) t} u_1
\]

Since the order of the equation (1) is finite, therefore, it is possible to choose \( A_1 > A_2 > A_3 > A_4 > A_5 \). Therefore, in order to solve equation (7) for the unknown functions \( A_1, A_2, A_3, A_4, A_5 \) and \( u_1 \), it is assumed that \( u_1 \) does not contain terms \( e^{-(i_1 A_1 + \cdots + i_5 A_5) t} \), where \( i_1 \leq 1 \) (see also \[^3\] for details). This is a significant assumption, since, under this assumption the coefficients of the terms of \( u_1 \) do not become large as well as \( u_1 \) does not contain secular type terms \( t e^{-\lambda t} \). Thus, in accordance with this assumptions, we obtain:

\[
e^{-A_1 t} \left( \frac{d}{dt} + A_2 - A_1 \right) \left( \frac{d}{dt} + A_3 - A_1 \right) \left( \frac{d}{dt} + A_4 - A_1 \right) \left( \frac{d}{dt} + A_5 - A_1 \right) u_1
\]

\[
+ e^{-A_1 t} \left( \frac{d}{dt} + A_1 - A_2 \right) \left( \frac{d}{dt} + A_3 - A_2 \right) \left( \frac{d}{dt} + A_4 - A_2 \right) \left( \frac{d}{dt} + A_5 - A_2 \right) u_2
\]

\[
+ e^{-A_1 t} \left( \frac{d}{dt} + A_1 - A_3 \right) \left( \frac{d}{dt} + A_2 - A_3 \right) \left( \frac{d}{dt} + A_4 - A_3 \right) \left( \frac{d}{dt} + A_5 - A_3 \right) u_3
\]

\[
+ e^{-A_1 t} \left( \frac{d}{dt} + A_1 - A_4 \right) \left( \frac{d}{dt} + A_2 - A_4 \right) \left( \frac{d}{dt} + A_3 - A_4 \right) \left( \frac{d}{dt} + A_5 - A_4 \right) u_4
\]

\[
+ e^{-A_1 t} \left( \frac{d}{dt} + A_1 - A_5 \right) \left( \frac{d}{dt} + A_2 - A_5 \right) \left( \frac{d}{dt} + A_3 - A_5 \right) \left( \frac{d}{dt} + A_4 - A_5 \right) u_5
\]

\[
= - \sum_{i_1=0}^{\infty} \cdots \sum_{i_5=0}^{\infty} F_{i_1} \cdots i_5(a_1, a_2, a_3, a_4, a_5) e^{-(i_1 A_1 + \cdots + i_5 A_5) t} u_1
\]

where \( i_1 \leq 1 \).

And:

\[
\left( \frac{d}{dt} + A_1 \right) \left( \frac{d}{dt} + A_2 \right) \left( \frac{d}{dt} + A_3 \right) \left( \frac{d}{dt} + A_4 \right) \left( \frac{d}{dt} + A_5 \right) u_1
\]

\[
= - \sum_{i_1=0}^{\infty} \cdots \sum_{i_5=0}^{\infty} F_{i_1} \cdots i_5(a_1, a_2, a_3, a_4, a_5) e^{-(i_1 A_1 + \cdots + i_5 A_5) t} u_1
\]

where:

\[
i_1 > 1.
\]
Now, we have to resolve the equation (8) for obtaining the unknown functions $A_1$, $A_2$, $A_3$, $A_4$ and $A_5$. However, the unknown functions $A_1$, $A_2$, $A_3$, $A_4$ and $A_5$ cannot be found easily by solving equation (8) if the nonlinear function $f$, and the eigenvalues, $-\lambda_1$, $-\lambda_2$, $-\lambda_3$, $-\lambda_4$ and $-\lambda_5$ are not specified. Therefore, to resolve the equation (8) for the unknown functions $A_1$, $A_2$, $A_3$, $A_4$, and $A_5$ and $u_1$, in this article, we have imposed the condition $\lambda_i \approx 3 \lambda_{i+1}$ ($i = 1, 2, 3, 4$) among the eigenvalues, i.e. the eigenvalues are multiple to one another. This condition is important, since such type of relation ($\lambda_i \approx 3 \lambda_{i+1}$), among the eigenvalues comes out naturally in the nonlinear equation with cubic nonlinearity. If the nonlinearity is the form other than cubic, in that case the condition $\lambda_i \approx 3 \lambda_{i+1}$ is not applicable and in that case different conditions should be used, and it is our work in the future. Using the above condition, we obtain:

$$
\begin{align*}
A_1 &= \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \ldots l_{i_1 \ldots i_n} e^{-(i_1 + i_2 + \ldots + i_n) t}, \\
A_2 &= \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \ldots m_{i_1 \ldots i_n} e^{-(i_1 + i_2 + \ldots + i_n) t}, \\
A_3 &= \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \ldots n_{i_1 \ldots i_n} e^{-(i_1 + i_2 + \ldots + i_n) t}, \\
A_4 &= \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \ldots p_{i_1 \ldots i_n} e^{-(i_1 + i_2 + \ldots + i_n) t}, \\
A_5 &= \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \ldots q_{i_1 \ldots i_n} e^{-(i_1 + i_2 + \ldots + i_n) t}, \\
\end{align*}
$$

(11)

where the coefficients $l_{i_1 \ldots i_n}$, $m_{i_1 \ldots i_n}$, $n_{i_1 \ldots i_n}$, $p_{i_1 \ldots i_n}$ and $q_{i_1 \ldots i_n}$ do not become large, as well as $A_1$, $A_2$, $A_3$, $A_4$ and $A_5$ do not become large, for any time $t$. Substituting the values of $A_1$, $A_2$, $A_3$, $A_4$ and $A_5$ from equation (11) into equation (4), and integrating, we shall obtain the values of $a_1$, $a_2$, $a_3$, $a_4$ and $a_5$. Again, solving equation (9), we shall obtain the value of $u_1$.

Thus, the determination of the solution is completed.

2. EXAMPLE

As an illustration of the above method, we have considered the Duffing equation type fifth order nonlinear differential systems:

$$
\frac{d^5x}{dt^5} + k_1 \frac{d^4x}{dt^4} + k_2 \frac{d^3x}{dt^3} + k_3 \frac{d^2x}{dt^2} + k_4 \frac{dx}{dt} + k_5 x = -\varepsilon x^3
$$

(12)

Here $f = x^3$, and thus, the system (12) is symmetrical (equation (12) remain unchanged if $x$ is replaced by $-x$).

Therefore,

$$
\begin{align*}
f^{(5)} &= a_1^3 e^{-3 \lambda_1 t} + 3 a_1^2 a_2 e^{-(2 \lambda_1 + \lambda_2) t} + 3 a_1 a_3 e^{-(3 \lambda_1 + 2 \lambda_2) t} + a_3^2 e^{-3 \lambda_2 t} + 3 a_1^2 a_3 e^{-(2 \lambda_1 + \lambda_3) t} \\
&+ 3 a_1 a_4 e^{-(2 \lambda_1 + \lambda_4) t} + 3 a_1^2 a_5 e^{-(2 \lambda_1 + \lambda_5) t} + 6 a_1 a_2 a_4 e^{-(\lambda_1 + \lambda_2 + \lambda_3) t} + 6 a_1 a_2 a_5 e^{-(\lambda_1 + \lambda_2 + \lambda_4) t} \\
&+ 6 a_1 a_3 a_4 e^{-(\lambda_1 + \lambda_3 + \lambda_4) t} + 6 a_1 a_3 a_5 e^{-(\lambda_1 + \lambda_3 + \lambda_5) t} + 3 a_1^2 a_4 e^{-(2 \lambda_1 + \lambda_2) t} + 3 a_1^2 a_5 e^{-(2 \lambda_1 + \lambda_3) t} \\
&+ 3 a_1 a_2 a_6 e^{-(\lambda_1 + \lambda_2 + \lambda_6) t} + 3 a_1 a_2 a_7 e^{-(\lambda_1 + \lambda_2 + \lambda_7) t} + 3 a_1 a_3 a_6 e^{-(\lambda_1 + \lambda_3 + \lambda_6) t} + 3 a_1 a_3 a_7 e^{-(\lambda_1 + \lambda_3 + \lambda_7) t} \\
&+ 3 a_1 a_4 a_5 e^{-(\lambda_1 + \lambda_4 + \lambda_5) t} + 3 a_1 a_4 a_6 e^{-(\lambda_1 + \lambda_4 + \lambda_6) t} + 3 a_1 a_4 a_7 e^{-(\lambda_1 + \lambda_4 + \lambda_7) t} + 3 a_1 a_5 a_7 e^{-(\lambda_1 + \lambda_5 + \lambda_7) t} \\
&+ 3 a_1^2 a_8 e^{-(2 \lambda_1 + \lambda_2 + \lambda_8) t} + 3 a_1 a_2 a_9 e^{-(\lambda_1 + \lambda_2 + \lambda_9) t} + 3 a_1^2 a_7 e^{-(\lambda_1 + 2 \lambda_7) t} + 3 a_1^2 a_8 e^{-(2 \lambda_1 + 2 \lambda_8) t} \\
&+ 3 a_1 a_2 a_10 e^{-(\lambda_1 + 2 \lambda_2 + \lambda_{10}) t} + 3 a_1 a_2 a_11 e^{-(\lambda_1 + 2 \lambda_2 + \lambda_{11}) t} + 3 a_1 a_2 a_12 e^{-(\lambda_1 + 2 \lambda_2 + \lambda_{12}) t} + 3 a_1^2 a_8 e^{-(\lambda_1 + 2 \lambda_8) t} \\
&+ 3 a_1 a_2 a_9 e^{-(\lambda_1 + \lambda_2 + \lambda_9) t} + 3 a_1^2 a_7 e^{-(\lambda_1 + 2 \lambda_7) t} + 3 a_1^2 a_8 e^{-(\lambda_1 + 2 \lambda_8) t} + 3 a_1^2 a_9 e^{-(\lambda_1 + 2 \lambda_9) t} + 3 a_1^2 a_10 e^{-(\lambda_1 + 2 \lambda_{10}) t} + 3 a_1^2 a_11 e^{-(\lambda_1 + 2 \lambda_{11}) t} + 3 a_1^2 a_{12} e^{-(\lambda_1 + 2 \lambda_{12}) t} + a_1^3 e^{-3 \lambda_1 t}
\end{align*}
$$

(13)
Therefore, for equation (12), equations (8)-(9) respectively become:

\[
\begin{align*}
&e^{-\lambda_1 t} \left( \frac{d}{dt} + \lambda_2 - \lambda_1 \right) \left( \frac{d}{dt} + \lambda_3 - \lambda_1 \right) \left( \frac{d}{dt} + \lambda_4 - \lambda_1 \right) \left( \frac{d}{dt} + \lambda_5 - \lambda_1 \right) A_1 \\
+ & e^{-\lambda_1 t} \left( \frac{d}{dt} + \lambda_1 - \lambda_2 \right) \left( \frac{d}{dt} + \lambda_3 - \lambda_2 \right) \left( \frac{d}{dt} + \lambda_4 - \lambda_2 \right) \left( \frac{d}{dt} + \lambda_5 - \lambda_2 \right) A_2 \\
+ & e^{-\lambda_1 t} \left( \frac{d}{dt} + \lambda_1 - \lambda_3 \right) \left( \frac{d}{dt} + \lambda_2 - \lambda_3 \right) \left( \frac{d}{dt} + \lambda_4 - \lambda_3 \right) \left( \frac{d}{dt} + \lambda_5 - \lambda_3 \right) A_3 \\
+ & e^{-\lambda_1 t} \left( \frac{d}{dt} + \lambda_1 - \lambda_4 \right) \left( \frac{d}{dt} + \lambda_2 - \lambda_4 \right) \left( \frac{d}{dt} + \lambda_3 - \lambda_4 \right) \left( \frac{d}{dt} + \lambda_5 - \lambda_4 \right) A_4 \\
+ & e^{-\lambda_1 t} \left( \frac{d}{dt} + \lambda_1 - \lambda_5 \right) \left( \frac{d}{dt} + \lambda_2 - \lambda_5 \right) \left( \frac{d}{dt} + \lambda_3 - \lambda_5 \right) \left( \frac{d}{dt} + \lambda_4 - \lambda_5 \right) A_5 \\
\end{align*}
\]

\[
= -[3a_1a_2^2e^{-(\lambda_1+\lambda_2)t} + 6a_1a_2a_3e^{-(\lambda_1+\lambda_2+\lambda_3)t} + 3a_1a_2^2e^{-(\lambda_1+\lambda_2)t} + 3a_1a_2^2e^{-(\lambda_1+\lambda_2)t} + 3a_1a_2^2e^{-(\lambda_1+\lambda_2)t} + 3a_1a_2^2e^{-(\lambda_1+\lambda_2)t} + 3a_1a_2^2e^{-(\lambda_1+\lambda_2)t} + 3a_1a_2^2e^{-(\lambda_1+\lambda_2)t} + 3a_1a_2^2e^{-(\lambda_1+\lambda_2)t}]
\]

\[
\]

\[
(14)
\]

And:

\[
\begin{align*}
& \left( \frac{d}{dt} + \lambda_1 \right) \left( \frac{d}{dt} + \lambda_2 \right) \left( \frac{d}{dt} + \lambda_3 \right) \left( \frac{d}{dt} + \lambda_4 \right) \left( \frac{d}{dt} + \lambda_5 \right) u_1 \\
& = - (a_1^3e^{-3\lambda_1t} + 3a_1^2a_2e^{-(2\lambda_1+\lambda_2)t} + 3a_1^2a_2e^{-(2\lambda_1+\lambda_2)t} + 3a_1^2a_2e^{-(2\lambda_1+\lambda_2)t} + 3a_1^2a_2e^{-(2\lambda_1+\lambda_2)t} + 3a_1^2a_2e^{-(2\lambda_1+\lambda_2)t})
\end{align*}
\]

\[
(15)
\]

where \( \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > \lambda_5 \). Solving equation (15), we obtain:

\[
\begin{align*}
& u_1 = r_1a_1^3e^{-3\lambda_1t} + r_2a_1^2a_2e^{-(2\lambda_1+\lambda_2)t} + r_3a_1^2a_2e^{-(2\lambda_1+\lambda_2)t} + r_4a_1^2a_2e^{-(2\lambda_1+\lambda_2)t} + r_5a_1^2a_2e^{-(2\lambda_1+\lambda_2)t}
\end{align*}
\]

\[
(16)
\]

where:

\[
\begin{align*}
& r_1 = 1/[(2\lambda_1(\lambda_2 - 3\lambda_1)(\lambda_3 - 3\lambda_1)(\lambda_4 - 3\lambda_1)(\lambda_5 - 3\lambda_1)) \\
& r_2 = 1/[((\lambda_1 + \lambda_3)(\lambda_4 - 3\lambda_1)(\lambda_5 - 3\lambda_1)) \\
& r_3 = 1/[((\lambda_1 + \lambda_3)(\lambda_4 - 3\lambda_1)(\lambda_5 - 3\lambda_1)) \\
& r_4 = 1/[((\lambda_1 + \lambda_3)(\lambda_4 - 3\lambda_1)(\lambda_5 - 3\lambda_1)) \\
& r_5 = 3/((\lambda_1 + \lambda_3)(\lambda_4 - 3\lambda_1)(\lambda_5 - 3\lambda_1))
\end{align*}
\]

\[
(17)
\]

For obtaining the unknown functions \( A_1, A_2, A_3, A_4, A_5 \), we have to separate the equation (14). Under the imposed condition (in this paper), we have imposed the condition \( \lambda_i \approx 3\lambda_{i+1}, \ i = 1, 2, 3, 4 \), we obtain:

\[
\begin{align*}
&e^{-\lambda_1 t} \left( \frac{d}{dt} - \lambda_1 + \lambda_2 \right) \left( \frac{d}{dt} - \lambda_1 + \lambda_3 \right) \left( \frac{d}{dt} - \lambda_1 + \lambda_4 \right) \left( \frac{d}{dt} - \lambda_1 + \lambda_5 \right) A_1 \\
= & - (3a_1a_2^2e^{-(\lambda_1+\lambda_2)t} + 6a_1a_2a_3e^{-(\lambda_1+\lambda_2+\lambda_3)t} + 3a_1a_2^2e^{-(\lambda_1+\lambda_2)t} + 3a_1a_2^2e^{-(\lambda_1+\lambda_2)t} + 3a_1a_2^2e^{-(\lambda_1+\lambda_2)t} + 3a_1a_2^2e^{-(\lambda_1+\lambda_2)t} + 3a_1a_2^2e^{-(\lambda_1+\lambda_2)t} + 3a_1a_2^2e^{-(\lambda_1+\lambda_2)t})
\end{align*}
\]

\[
(18)
\]

\[
\begin{align*}
&e^{-\lambda_1 t} \left( \frac{d}{dt} - \lambda_2 + \lambda_1 \right) \left( \frac{d}{dt} - \lambda_2 + \lambda_3 \right) \left( \frac{d}{dt} - \lambda_2 + \lambda_4 \right) \left( \frac{d}{dt} - \lambda_2 + \lambda_5 \right) A_2 \\
= & - (3a_2^3e^{-(2\lambda_1+\lambda_2)t} + 3a_2^3e^{-(2\lambda_1+\lambda_2)t} + 3a_2^3e^{-(2\lambda_1+\lambda_2)t} + 3a_2^3e^{-(2\lambda_1+\lambda_2)t} + 3a_2^3e^{-(2\lambda_1+\lambda_2)t} + 6a_2a_3a_5e^{-(\lambda_2+\lambda_4+\lambda_5)t} + a_2^3e^{-3\lambda_2t})
\end{align*}
\]

\[
(19)
\]
Therefore, solving equations (18)-(22), we obtain:

\[ A_1 = l_1 a_1 a_2^2 e^{-2l_1t} + l_2 a_1 a_2 a_3 e^{-(l_1 \pm l_4) t} + l_3 a_1 a_2 a_3 e^{-2l_1t} + l_4 a_1 a_2 e^{-2l_1t} + l_5 a_1 a_2^2 e^{-2l_1t} + l_6 a_1 a_2 a_3 e^{-(l_1 + l_4) t} + l_7 a_1 a_2 a_3 e^{-(l_1 + l_2 + l_4) t} + l_8 a_1 a_3 a_5 e^{-(l_1 + l_5) t} + l_9 a_1 a_2 a_4 e^{-(l_1 + l_3) t} + l_{10} a_1 a_2 a_5 e^{-(l_1 + l_3) t} + l_{11} a_2^2 e^{(l_1 - 3l_3) t} \]

(23)

\[ A_2 = m_1 a_1 a_3 e^{-(l_1 + l_4) t} + m_2 a_1 a_4 e^{-(l_1 + l_4) t} + m_3 a_2 a_5 e^{-(l_1 + l_4) t} + m_4 a_2 a_3 e^{-2l_1t} + m_5 a_1 a_2 e^{-2l_1t} + m_6 a_2 a_3 a_4 e^{-(l_1 + l_4) t} + m_7 a_2 a_3 a_4 e^{-(l_1 + l_3) t} + m_8 a_2 a_3 a_4 e^{-(l_1 + l_4) t} + m_9 a_2 a_3 a_4 e^{-(l_1 + l_3) t} + m_{10} a_2^3 e^{(l_1 - 3l_3) t} \]

(24)

\[ A_3 = n_1 a_1 a_3 e^{-(l_1 + l_4) t} + n_2 a_1 a_3 e^{-(l_1 + l_4) t} + n_3 a_2 a_3 e^{-2l_1t} + n_4 a_2 a_3 e^{-(l_1 + l_4) t} + n_5 a_2 a_3 e^{-(l_1 + l_3) t} + n_6 a_2 a_3 e^{-(l_1 + l_4) t} \]

(25)

\[ A_4 = p_1 a_1 a_3 e^{-(l_1 - 3l_3) t} \]

(26)

\[ A_5 = 0 \]

(27)

where:

\[ l_1 = 3/(a_1 + a_2)(a_3 - a_2 - 2a_3)(a_4 - a_2 - 2a_4)(a_5 - a_2 - 2a_5) \]

\[ l_2 = -6/(a_1 + a_3)(a_4 - a_1 - a_3 - a_5)(a_5 - a_1 - a_3 - a_5) \]

\[ l_3 = 3/(a_1 + a_3)(a_2 - a_1 - 2a_3)(a_4 - a_3 - 2a_4)(a_5 - a_1 - 2a_5) \]

\[ l_4 = 3/(a_1 + a_4)(a_2 - a_1 - 2a_4)(a_3 - a_1 - 2a_3)(a_4 - a_1 - 2a_4) \]

\[ l_5 = 3/(a_1 + a_5)(a_2 - a_1 - 2a_5)(a_3 - a_1 - 2a_3)(a_4 - a_1 - 2a_4) \]

\[ l_6 = -6/(a_1 + a_3)(a_2 - a_1 - 2a_4)(a_3 - a_1 - 2a_3)(a_4 - a_1 - 2a_4) \]

\[ l_7 = -6/(a_1 + a_4)(a_3 - a_1 - 2a_3)(a_4 - a_1 - 2a_4) \]

\[ l_8 = -6/(a_1 + a_5)(a_3 - a_1 - 2a_3) \]

\[ l_9 = -6/(a_1 + a_5)(a_3 - a_1 - 2a_3)(a_4 - a_1 - 2a_4) \]

\[ l_{10} = -6/(a_1 + a_5)(a_3 - a_1 - 2a_3)(a_4 - a_1 - 2a_4) \]

\[ l_{11} = 1/[2(a_2)(a_5 - a_2 - a_5)(a_4 - a_2 - a_5) \]

\[ m_1 = 3/(2a_2)(a_1 - 2a_2 - a_5)(a_4 - 2a_2 - a_5) \]

\[ m_2 = 3/(2a_2)(a_1 - 2a_2 - a_5)(a_3 - 2a_2 - a_5) \]

\[ m_3 = 3/(2a_2)(a_1 - 2a_2 - a_5)(a_3 - 2a_2 - a_5)(a_4 - 2a_2 - a_5) \]
Substituting the values of $A_1, A_2, A_3, A_4$ and $A_5$ from equations (23)-(27) into equation (4), we obtain:

$$
\frac{da_1}{dt} = \varepsilon \left( l_1 a_1 a_2^2 e^{-2\lambda t} + l_2 a_1 a_2 a_3 e^{-(\lambda_1+\lambda_2)t} + l_3 a_1 a_3^2 e^{-2\lambda t} + l_4 a_1 a_2^2 e^{-2\lambda t} + l_5 a_1 a_2^2 e^{-2\lambda t} + l_6 a_1 a_2 a_3 e^{-(\lambda_1+\lambda_2)t} + l_7 a_1 a_3 a_5 e^{-(\lambda_1+\lambda_2)t} + l_8 a_1 a_2 a_4 e^{-(\lambda_1+\lambda_2)t} + l_9 \frac{a_3}{a_2} e^{-(\lambda_1+\lambda_2)t} + l_{10} \frac{a_1}{a_2} e^{-(\lambda_1+\lambda_2)t} \right) \tag{28}
$$

$$
\frac{da_2}{dt} = \varepsilon \left( m_1 a_2 a_3 a_5 e^{-(\lambda_1+\lambda_2)t} + m_2 a_2 a_4 e^{-(\lambda_1+\lambda_2)t} + m_3 a_2 a_5 e^{-(\lambda_1+\lambda_2)t} + m_4 a_2 a_3 e^{-(\lambda_1+\lambda_2)t} + m_5 a_2 a_3 e^{-(\lambda_1+\lambda_2)t} + m_6 a_2 a_3 e^{-(\lambda_1+\lambda_2)t} + m_7 a_2 a_3 e^{-(\lambda_1+\lambda_2)t} + m_8 a_2 a_3 e^{-(\lambda_1+\lambda_2)t} + m_9 a_2 a_3 e^{-(\lambda_1+\lambda_2)t} + m_{10} a_2 a_3 e^{-(\lambda_1+\lambda_2)t} \right)
$$

$$
\frac{da_3}{dt} = \varepsilon \left( n_1 a_3 a_2 e^{-(\lambda_1+\lambda_2)t} + n_2 a_3 a_4 e^{-(\lambda_1+\lambda_2)t} + n_3 a_3 a_5 e^{-(\lambda_1+\lambda_2)t} + n_4 a_3 a_6 e^{-(\lambda_1+\lambda_2)t} + n_5 a_3 a_7 e^{-(\lambda_1+\lambda_2)t} + n_6 a_3 a_8 e^{-(\lambda_1+\lambda_2)t} + n_7 a_3 a_9 e^{-(\lambda_1+\lambda_2)t} + n_8 a_3 a_10 e^{-(\lambda_1+\lambda_2)t} \right)
$$

Since $\varepsilon$ is a small quantity, we can solve (28) by assuming that $a_1, a_2, a_3, a_4$ and $a_5$ are constants in the right-hand side of (28). This assumption was first made by Murty et al.\cite{11,12} to solve similar type of nonlinear equations. Thus the solutions of the equations of (28) are:

$$
a_1(t) = a_{1,0} + \varepsilon \left[ l_1 a_{1,0} a_2^2 + \frac{1}{2} \frac{a_2}{a_1} e^{-2\lambda t} + l_2 a_{1,0} a_2 a_3 + \frac{1}{2} \frac{a_3}{a_1} e^{-(\lambda_1+\lambda_2)t} + l_3 a_{1,0} a_3^2 + \frac{1}{2} \frac{a_3}{a_1} e^{-2\lambda t} + l_4 a_{1,0} a_2^2 + \frac{1}{2} \frac{a_2}{a_1} e^{-2\lambda t} + l_5 a_{1,0} a_2^2 + \frac{1}{2} \frac{a_2}{a_1} e^{-2\lambda t} + l_6 a_{1,0} a_2 a_3 + \frac{1}{2} \frac{a_3}{a_1} e^{-(\lambda_1+\lambda_2)t} + l_7 a_{1,0} a_3 a_5 + \frac{1}{2} \frac{a_5}{a_1} e^{-(\lambda_1+\lambda_2)t} + l_8 a_{1,0} a_2 a_4 + \frac{1}{2} \frac{a_4}{a_1} e^{-(\lambda_1+\lambda_2)t} + l_9 \frac{a_3}{a_2} + \frac{1}{2} \frac{a_2}{a_1} e^{-(\lambda_1+\lambda_2)t} + l_{10} \frac{a_1}{a_2} + \frac{1}{2} \frac{a_2}{a_1} e^{-(\lambda_1+\lambda_2)t} \right]
$$

$$
a_2(t) = a_{2,0} + \varepsilon \left[ m_1 a_{2,0} a_3 a_5 + \frac{1}{2} \frac{a_5}{a_2} e^{-(\lambda_1+\lambda_2)t} + m_2 a_{2,0} a_4 + \frac{1}{2} \frac{a_4}{a_2} e^{-(\lambda_1+\lambda_2)t} + m_3 a_{2,0} a_5 + \frac{1}{2} \frac{a_5}{a_2} e^{-(\lambda_1+\lambda_2)t} + m_4 a_{2,0} a_3 + \frac{1}{2} \frac{a_3}{a_2} e^{-(\lambda_1+\lambda_2)t} + m_5 a_{2,0} a_3 + \frac{1}{2} \frac{a_3}{a_2} e^{-(\lambda_1+\lambda_2)t} + m_6 a_{2,0} a_3 + \frac{1}{2} \frac{a_3}{a_2} e^{-(\lambda_1+\lambda_2)t} + m_7 a_{2,0} a_3 + \frac{1}{2} \frac{a_3}{a_2} e^{-(\lambda_1+\lambda_2)t} + m_8 a_{2,0} a_3 + \frac{1}{2} \frac{a_3}{a_2} e^{-(\lambda_1+\lambda_2)t} + m_9 a_{2,0} a_3 + \frac{1}{2} \frac{a_3}{a_2} e^{-(\lambda_1+\lambda_2)t} + m_{10} a_{2,0} a_3 + \frac{1}{2} \frac{a_3}{a_2} e^{-(\lambda_1+\lambda_2)t} \right]
$$

$$
a_3(t) = a_{3,0} + \varepsilon \left[ n_1 a_{3,0} a_2 + \frac{1}{2} \frac{a_2}{a_3} e^{-(\lambda_1+\lambda_2)t} + n_2 a_{3,0} a_4 + \frac{1}{2} \frac{a_4}{a_3} e^{-(\lambda_1+\lambda_2)t} + n_3 a_{3,0} a_5 + \frac{1}{2} \frac{a_5}{a_3} e^{-(\lambda_1+\lambda_2)t} + n_4 a_{3,0} a_6 + \frac{1}{2} \frac{a_6}{a_3} e^{-(\lambda_1+\lambda_2)t} + n_5 a_{3,0} a_7 + \frac{1}{2} \frac{a_7}{a_3} e^{-(\lambda_1+\lambda_2)t} + n_6 a_{3,0} a_8 + \frac{1}{2} \frac{a_8}{a_3} e^{-(\lambda_1+\lambda_2)t} + n_7 a_{3,0} a_9 + \frac{1}{2} \frac{a_9}{a_3} e^{-(\lambda_1+\lambda_2)t} + n_8 a_{3,0} a_{10} + \frac{1}{2} \frac{a_{10}}{a_3} e^{-(\lambda_1+\lambda_2)t} \right]
$$
Therefore, we have considered the eigenvalues \( \lambda \) and \( \lambda_2 \). Secondly, we have considered the eigenvalues \( \lambda \) and \( \lambda_2 \).

First of all, we have considered the eigenvalues \( \lambda \) and \( \lambda_2 \). Then, we refer to the work of Murty \[12\] to compare the approximate solution to the numerical solution. With regard to such a comparison concerning \( \lambda \) and \( \lambda_2 \), we proceed as follows:

\[
a_1(t) = a_{20} + \epsilon \left[ m_1 a_{2,0}^a a_{1,0} - e^{-\lambda_1 t} + m_2 a_{2,0}^a a_{4,0} - e^{-\lambda_2 t} + m_3 a_{2,0}^a a_{5,0} - e^{-\lambda_3 t} + m_4 a_{2,0}^a a_{4,0} - e^{-\lambda_4 t} + m_5 a_{2,0}^a a_{5,0} - e^{-\lambda_5 t} + m_6 a_{2,0}^a a_{6,0} - e^{-\lambda_6 t} + m_7 a_{2,0}^a a_{7,0} - e^{-\lambda_7 t} \right]
\]

\[
a_2(t) = a_{3,0} + \epsilon \left[ n_1 a_{3,0}^a - e^{-\lambda_3 t} + n_2 a_{3,0}^a a_{4,0} - e^{-\lambda_4 t} + n_3 a_{3,0}^a a_{5,0} - e^{-\lambda_5 t} + n_4 a_{3,0}^a a_{6,0} - e^{-\lambda_6 t} + n_5 a_{3,0}^a a_{7,0} - e^{-\lambda_7 t} \right]
\]

\[
a_3(t) = a_{4,0} + \epsilon \left[ p_1 a_{4,0}^a - e^{-\lambda_4 t} + p_2 a_{4,0}^a a_{5,0} - e^{-\lambda_5 t} + p_3 a_{4,0}^a a_{6,0} - e^{-\lambda_6 t} + p_4 a_{4,0}^a a_{7,0} - e^{-\lambda_7 t} \right]
\]

\[
a_4(t) = a_{5,0}
\]

when \( \lambda_{i1} \approx 3 \lambda_{i1+1} \), \( i = 1, 2, 3, 4 \). But when \( \lambda_{i1} = 3 \lambda_{i1+1} \), \( i = 1, 2, 3, 4 \), the terms \( l_{11} a_{20}^a - e^{-\lambda_1 t} \) of the first, second, third and fourth equation of (29) will be replaced by \( l_{11} a_{20}^a - e^{-\lambda_1 t} \), \( m_{11} a_{3,0}^a - e^{-\lambda_3 t} \), \( n_{11} a_{4,0}^a - e^{-\lambda_4 t} \) and \( p_{11} a_{5,0}^a - e^{-\lambda_5 t} \) respectively.

Therefore, the first-order approximate solution of the equation (12) is:

\[
x(t, \epsilon) = a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t} + a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t} + a_5 e^{-\lambda_5 t} + \epsilon u_1
\]

where \( a_1, a_2, a_3, a_4 \) and \( a_5 \) are given by equation (29) and \( u_1 \) is given by equation (16).

3. RESULTS AND DISCUSSION

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, we compare the approximate solution to the numerical solution. With regard to such a comparison concerning the presented technique of this article, we refer to the work of Murty \textit{et al.}\[12\].

First of all, we have considered the eigenvalues \( \lambda_1 \approx 42.0, \lambda_2 \approx 13.60, \lambda_3 \approx 4.55, \lambda_4 \approx 1.6, \lambda_5 \approx 0.5 \). Therefore \( \lambda_{i1} \approx 3 \lambda_{i1+1} \), \( i = 1, 2, 3, 4 \). We have computed \( x(t, \epsilon) \) by equation (30) in which \( a_1, a_2, a_3, a_4 \) and \( a_5 \) are computed by equation (29) together with initial conditions \( a_{1,0} = 0.25, a_{2,0} = 0.25, a_{3,0} = 0.25, a_{4,0} = 0.25 \) and \( a_{5,0} = 0.25 \) for various values of \( \epsilon \) when \( \epsilon = 0.1 \). The perturbation results are presented in Figure 1, plotted by the dot line. The corresponding numerical solution has been computed by a fourth-order Runge-Kutta method, and the results are presented in Figure 1, plotted by the continuous line. The correlation between these two results has also been calculated which is 0.999505.

Secondly, we have considered the eigenvalues \( \lambda_1 \approx 52.0, \lambda_2 \approx 17.30, \lambda_3 \approx 5.70, \lambda_4 \approx 1.92, \lambda_5 \approx 0.63 \). Therefore \( \lambda_{i1} \approx 3 \lambda_{i1+1} \), \( i = 1, 2, 3, 4 \). We have computed \( x(t, \epsilon) \) using equation (30) in which \( a_1, a_2, a_3, a_4 \) and \( a_5 \) are computed by equation (29), together with initial conditions \( a_{1,0} = 0.28, a_{2,0} = 0.32, a_{3,0} = 0.32, a_{4,0} = 0.40 \) and \( a_{5,0} = 0.40 \) for various values of \( \epsilon \) when \( \epsilon = 0.25 \) and the perturbation...
results are presented in Figure 2, plotted by the dot line. The corresponding numerical solution has been computed by a fourth-order Runge-Kutta method, and the results are presented in Figure 2, plotted by the continuous line. The correlation between these two results have also been calculated which is 0.997105.

Finally, we have considered the eigenvalues \( \lambda_1 = 21.30, \lambda_2 = 7.09, \lambda_3 = 2.35, \lambda_4 = 0.77, \lambda_5 = 0.250 \). Therefore \( \lambda_i \approx 3\lambda_{i+1}, i = 1, 2, 3, 4 \). We have again computed \( x(t, \varepsilon) \) by equation (30) in which \( a_1, a_2, a_3, a_4 \) and \( a_5 \) are computed by equation (29), together with initial conditions \( a_{1,0} = 0.25, a_{2,0} = 0.25, a_{3,0} = 0.25, \)
$a_{4,0} = 0.25$ and $a_{5,0} = 0.25$ \[ \text{or } x(0) = 2.100000, \quad \frac{dx(0)}{dt} = -13.568513, \quad \frac{d^2x(0)}{dt^2} = 222.005787, \quad \frac{d^3x(0)}{dt^3} = -4445.7000619 \quad \text{and} \quad \frac{d^4x(0)}{dt^4} = 94031.172125 \] for various values of $t$ when $\epsilon = 0.1$. The perturbation results are presented in Figure 3, plotted by the dot line. The corresponding numerical solution has been computed by a fourth-order Runge-Kutta method, and the results are presented in Figure 3, plotted by continuous line. The correlation between these two results have also been calculated which is 0.999995.

Figure 3
Perturbation Results are Plotted by Dot Line and Numerical Results are Plotted by Continuous Line

CONCLUSION

A perturbation technique, based on the work of Akbar et al.\(^3\) is developed to obtain the transient response of fifth-order over-damped symmetrical nonlinear systems. We calculated the correlation between the results obtained by the presented perturbation technique and the corresponding numerical results obtained by the fourth-order Runge-Kutta method. These two results are strongly-correlated. The results obtained for different sets of initial conditions, as well as for different damping forces, show good coincidence with those results obtained by numerical method.

REFERENCES


