Algebraic Properties of the Category of Q-P Quantale Modules

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Abstract: In this paper, the definition of a Q-P quantale module and some relative concepts were introduced. Based on which, some properties of the Q-P quantale module, and the structure of the free Q-P quantale modules generated by a set were obtained. It was proved that the category of Q-P quantale modules is algebraic.

Key words: Q-P quantale quantale modules; Equalizer; Forgetful functor; Algebraic category

1. INTRODUCTION

Quantale was proposed by Mulvey in 1986 for studying the foundations of quantum logic and for studying non-commutation C*-algebras. The term quantale was coined as a combination of “quantum logic” and “locale” by Mulvey in [1]. The systematic introduction of quantale theory came from the book [2], which written by Rosenthal in 1990.
Since quantale theory provides a powerful tool in studying noncommutative structures, it has a wide applications, especially in studying noncommutative C*-algebra theory [3], the ideal theory of commutative ring [4], linear logic [5] and so on. So, the quantale theory has aroused great interests of many scholar and experts, a great deal of new ideas and applications of quantale have been proposed in twenty years [6–18].

Since the ideal of quantale module was proposed by Abramsky and Vickers [19], the quantale module has attracted many scholars eyes. With the development of the quantale theory, the theory of quantale module was studied deeply in [20–25]. In this paper, some properties of the category of Q-P quantale modules was discussed, especially that the category of Q-P quantale modules is algebraic was proved.

2. PRELIMINARIES

Definition 2.1 [2] A quantale is a complete lattice $Q$ with an associative binary operation "&" satisfying:

$$a \& (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \& b_i) \quad \text{and} \quad (\bigvee_{i \in I} b_i) \& a = \bigvee_{i \in I} (b_i \& a),$$

for all $a, b_i \in Q$, where $I$ is a set, 0 and 1 denote the smallest element and the greatest element of $Q$ respectively.

Definition 2.2 A nonzero element $a$ in a quantale $Q$ is said to be a nonzero divisor if for all nonzero element $b \in Q$ such that $a \& b \neq 0$, $b \& a \neq 0$. $Q$ is nonzero divisor if every $a \in Q$ is a nonzero divisor.

Definition 2.3 Let $Q, P$ be a quantale, a Q-P quantale module over $Q, P$ (briefly, a Q-P-module) is a complete lattice $M$, together with a mapping $T : Q \times M \times P \to M$ satisfies the following conditions:

1. $T(\bigvee_{i \in I} a_i, m, \bigvee_{j \in J} b_j) = \bigvee_{i \in I, j \in J} T(a_i, m, b_j)$;
2. $T(a, \bigvee_{k \in K} m_k, b) = \bigvee_{k \in K} T(a, m_k, b)$;
3. $T(a \& b, m, c \& d) = T(a, T(b, m, c), d)$ for all $a_i, a, b \in Q$, $b_j, c, d \in P$, $m_k, m \in M$. We shall denote the Q-P quantale module $M$ over $Q, P$ by $(M, T)$.

If $Q$ is unital quantale with unit $e$, we define $T(e, m, e) = m$ for all $m \in M$.

Example 2.4 (1) Let $Q = P = \{0, a, b, c, 1\}$ be a set, $M = \{0, d, e, 1\}$ is a complete lattice. The order relations of $Q$ and $M$ are given by the following figure 1 and 2, we give a binary operator "&" on $Q$ satisfying the diagram 1.

![Diagram 1](image1.png)

![Diagram 2](image2.png)

We can prove that $Q$ is a quantale.
Now, define a mapping $T : Q \times M \times Q \to M$ such that $T(x, m, y) = m$ for all $x, y \in Q$, $m \in M$. Then $(M, T)$ be a Q-P quantale module.

(2) Let $Q = P = \{0, a, b, 1\}$ be a complete lattice. The order relation on $Q$ satisfies the following Figure 3 and the binary operation of $Q$ satisfies the diagram 2:

$$
\begin{array}{cc|cccc}
& & 0 & a & b & 1 \\
0 & 0 & 0 & 0 & 0 & \\
a & 0 & a & 0 & a & \\
b & 0 & 0 & b & b & \\
1 & 0 & a & b & 1 & \\
\end{array}
$$

Figure 3

Diagram 2

It is easy to show that $(Q, \&)$ is a quantale. Let $M = \{0, a, 1\} \subseteq Q$, then $M$ is a complete lattice with the inheriting order on $Q$. Now, we define $T : Q \times M \times Q \to M$ is a Q-P quantale module.

**Definition 2.5** Let $Q, P$ be a quantale, $(M_1, T_1)$ and $(M_2, T_2)$ are Q-P quantale modules. A mapping $f : M_1 \to M_2$ is said to be a Q-P quantale module homomorphism if $f$ satisfies the following conditions:

1. $f(\bigvee_{i \in I} m_i) = \bigvee_{i \in I} f(m_i)$;
2. $f(T_1(a, m, b)) = T_2(a, f(m), b)$ for all $a \in Q, b \in P, m_i, m \in M$.

**Definition 2.6** Let $(M, T_M)$ be a $Q - P$ quantale module over $Q, P$, $N$ be a subset of $M$, $N$ is said to be a submodule of $M$ if $N$ is closed under arbitrary join and $T_M(a, n, b) \in N$ for all $a \in Q, b \in P, n \in N$.

**Definition 2.7** [26] A concrete category $(A, U)$ is called algebraic provided that it satisfies the following conditions:

1. $A$ has coequalizers;
2. $U$ has a left adjoint;
3. $U$ preserves and reflects regular epimorphisms.

### 3. THE CATEGORY OF Q-P QUANTALE MODULES IS ALGEBRAIC

**Definition 3.1.** Let $Q, P$ be a quantale, $\mathbf{QMod}_P$ be the category whose objects are the Q-P quantale modules of $Q, P$, and morphisms are the Q-P quantale module homomorphisms, i.e.,

- $\text{Ob}(\mathbf{QMod}_P) = \{ M : M \text{ is Q-P quantale modules} \}$,
- $\text{Mor}(\mathbf{QMod}_P) = \{ f : M \to N \text{ is the Q-P quantale modules homorphism} \}$.

Hence, the category $\mathbf{QMod}_P$ is a concrete category.

**Definition 3.2.** Let $Q, P$ is a quantale, $(M, T_M)$ is a Q-P quantale module, $R \subseteq M \times M$. The set $R$ is said to be a congruence of Q-P quantale module on $M$ if $R$ satisfies:

1. $R$ is an equivalence relation on $M$;
2. If $(m_i, n_i) \in R$ for all $i \in I$, then $(\bigvee_{i \in I} m_i, \bigvee_{i \in I} n_i) \in R$;
(3) If \((m, n) \in R\), then \((T_M(a, m, b), T_M(a, n, b)) \in R\) for all \(a \in Q, b \in P\).

We denote the set of all congruence on \(M\) by \(\text{Con}(Q M_P)\), then \(\text{Con}(Q M_P)\) is a complete lattice with respect to the inclusion order.

Let \(Q, P\) be a quantale, \(M\) is a Q-P quantale module, \(R\) is a congruence of Q-P quantale module on \(M\), define the order relation on \(M/R\) such that \([m] \leq [n]\) if and only if \([m \lor n] = [n]\) for all \([m], [n] \in M/R\).

**Theorem 3.3.** Let \(Q, P\) be a quantale, \(M\) be a Q-P quantale module, \(R\) be a congruence of double quantale module on \(M\). Define \(T_{M/R} : Q \times M/R \times P \longrightarrow M/R\) such that \(T_{M/R}(a, [m], b) = [T_M(a, m, b)]\) for all \(a \in Q, b \in P, [m] \in M/R\), then \((M/R, T_{M/R})\) is a Q-P quantale module and \(\pi : m \mapsto [m] : M \longrightarrow M/R\) is a Q-P quantale module homomorphisms.

**Proof.** (1) We will prove that \(\leq\) is a partial order on \(M/R\), and \(T_{M/R}\) is well defined. In fact, for all \([m], [n], [l] \in M/R\),

(i) It’s clearly that \([m] \leq [m]\);

(ii) Let \([m] \leq [n], [n] \leq [m]\), then \([m \lor n] = [n]\) and \([n \lor m] = [m]\), thus \([m] = [n]\);

(iii) Let \([m] \leq [n], [n] \leq [l]\), then \([m \lor n] = [n]\) and \([n \lor l] = [l]\), therefore \([m \lor l] = [m \lor (n \lor l)] = [m \lor n] \lor e = [n] \lor l = [l]\).

If \([m_1] = [m_2]\), then \([m_1, m_2] \in R, (T_M(a, m, b), T_M(a, n, b)) \in R\) for all \(a, b \in Q\), i.e., \([T_M(a, m, b)] = [T_M(a, n, b)]\), thus \(T_{M/R}\) is well defined.

(2) We will prove that \((M/R, \leq)\) is a complete lattice. Let \(\{m_i \mid i \in I\} \subseteq M/R\), we have

(i) Since \([m_i \lor \bigvee_{i \in I} m_i] = \bigvee_{i \in I} m_i\) for all \(i \in I\), then \([m_i] \leq \bigvee_{i \in I} m_i\);

(ii) Let \([m] \in M/R\) and \([m_i] \leq [m]\) for all \(i \in I\), then \([m_i \lor m] = [m]\) for all \(i \in I\), hence \([\bigvee_{i \in I} m_i \lor m] = \bigvee_{i \in I} (m_i \lor m) = [m]\), i.e., \(\bigvee_{i \in I} m_i \leq [m]\).

Thus \(\bigvee_{i \in I} [m_i] = [\bigvee_{i \in I} m_i]\).

(3) For all \(\{a_i \mid i \in I\} \subseteq Q, \{b_j \mid j \in J\} \subseteq Q, \{[m_i] \mid l \in H\} \subseteq M/R, a, b \in Q, c, d \in P\), we have that

(i) \(T_{M/R}\left(\bigvee_{i \in I} a_i, [m], \bigvee_{j \in J} b_j\right) = [T_M\left(\bigvee_{i \in I} a_i, m, \bigvee_{j \in J} b_j\right)] = \bigvee_{i \in I, j \in J} T_M(a_i, m, b_j)\)

(ii) \(T_{M/R}(a, \bigvee_{j \in J} m_j), b) = T_{M/R}(a, \bigvee_{j \in J} m_j, b) = [T_M(a, \bigvee_{j \in J} m_j, b)] = \bigvee_{j \in J} [T_M(a, m_j, b)]

(iii) \(T_{M/R}(a \land b, [m], c \land d) = [T_M(a \land b, m, c \land d)] = [T_M(a, \underline{T_M(b, m, c, d)})\]

Then \((M/R, T_{M/R})\) is a Q-P quantale module.

(4) For all \(\{[m_i] \mid i \in I\} \subseteq M/R, a \in Q, b \in P\), we have that

\(\pi(\bigvee_{i \in I} m_i) = [\bigvee_{i \in I} m_i] = \bigvee_{i \in I} [m_i] = \bigvee_{i \in I} \pi(m_i)\)

\(\pi(T_{M/R}(a, m, b)) = T_{M/R}(a, [m], b) = T_{M/R}(a, \pi(m), b)\).

So \(\pi : m \mapsto [m] : M \longrightarrow M/R\) is a Q-P quantale module homomorphisms.

**Theorem 3.4.** Let \(Q, P\) be a quantale, \(M\) a double quantale module, then \(\Delta = \{(x, x) \mid x \in M\}\) is a congruence of Q-P quantale module on \(M\).

**Theorem 3.5.** Let \(Q, P\) be a quantale, \(M\) and \(N\) be Q-P quantale modules, \(f : M \longrightarrow N\) a Q-P quantale module homorphism, \(R\) a Q-P quantale module.
congruence on $N$. Then $f^{-1}(R) = \{(x, y) \in M \times M \mid (f(x), f(y)) \in R \}$ is a Q-P quantale module congruence on $M$.

**Theorem 3.6.** Let $Q, P$ be a quantale, $M$ and $N$ are Q-P quantale modules, $f : M \rightarrow N$ be a Q-P quantale module homorphism. Then $f^{-1}(\Delta) = \{(x, y) \in M \times M \mid f(x) = f(y)\}$ be a Q-P quantale module congruence on $M$, where $\Delta = \{(a, a) \mid a \in N\}$.

Let $Q, P$ be a quantale, $M$ be a Q-P quantale module, $R \subseteq M \times M$, since $\text{Con}(QMP)$ is a complete lattice, there exists a smallest Q-P quantale congruence containing $R$, which is the intersection all the Q-P quantale module congruence containing $R$ on $M$. We said that this congruence is generated by $R$.

**Theorem 3.7.** The category $\mathcal{QMod}_P$ has coequalizer.

$$
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
g \downarrow & & \downarrow h \\
\pi & & E' \\
\downarrow \overline{h} & & \\
E & &
\end{array}
$$

**Proof.** Let $Q, P$ be a quantale, $(M, T_M)$ and $(N, T_N)$ be Q-P quantale modules, $f$ and $g$ be Q-P quantale module homomorphisms, Suppose $R$ is the smallest congruence of the Q-P quantale modules on $N$, which contain $\{(f(x), g(x)) \mid x \in M\}$. Let $E = N/R$, $\pi : N \rightarrow N/R$ is the canonical mapping, then $(N/R, T_{N/R})$ is a Q-P quantale module and $\pi$ is a Q-P quantale module homomorphisms by theorem 3.3. We will prove that $(\pi, E)$ is the coequalizer of $f$ and $g$. In fact,

1. $\pi \circ f = \pi \circ g$ is clear.

2. Let $(E', T_{E'})$ be a Q-P quantale module, $h : N \rightarrow E'$ be a Q-P quantale module homomorphisms such that $h \circ f = h \circ g$. Let $R_1 = h^{-1}(\Delta)$, where $\Delta = \{(x, x) \mid x \in E'\}$. By theorem 3.5, we can see that $R_1$ is a congruence of Q-P quantale module on $N$. Since $h(f(x)) = h(g(x))$ for all $x \in M$, then $(f(x), g(x)) \in R_1$. Define $\overline{h} : N/R \rightarrow E'$ such that $\overline{h}([n]) = h(n)$ for all $[n] \in N/R$. Let $n_1, n_2 \in N$ and $(n_1, n_2) \in R$, then $(n_1, n_2) \in R_1$, and we have that $h(n_1) = h(n_2)$. Therefore $\overline{h}$ is well defined.

For all $\{[n_i] \mid i \in I\} \subseteq N/R$, $a, b \in Q$, $[n] \in N/R$, we have that

$$
\overline{h}(\bigvee_{i \in I} [n_i]) = \overline{h}(\bigvee_{i \in I} n_i) = \bigvee_{i \in I} h(n_i) = \bigvee_{i \in I} \overline{h}([n_i]);
$$

$$
\overline{h}(T_{N/R}(a, [n], b)) = \overline{h}([T(a, n, b)]) = h(T(a, n, b)) = T_{E'}(a, h(n), b) = T_{E'}(a, \overline{h}([n]), b).
$$

Thus, $\overline{h}$ is a Q-P quantale module, and $\overline{h}$ is the unique homomorphism satisfy $\overline{h} \circ \pi = h$. Therefore $(\pi, E)$ is the coequalizer of $f$ and $g$. $\square$

From now until the end of Section 3, we suppose $Q$ be a unital quantale with unit $e$. Let $X$ be a nonempty set, we consider the complete lattice $(Q^X, \bigvee X)$, where $Q^X$ is the set of all the function from $X$ to $Q$ and $(\bigvee_{i \in I} f_i)(x) = \bigvee_{i \in I} f_i(x)$ for all $x \in X$.

**Theorem 3.8.** Let $X$ be a nonempty set, and $Q$ is idempotent and unital quantale with unit $e$, define $T_X : Q \times Q^X \times Q \rightarrow Q^X$ such that $T_X(a, f, b)(x) = a \& f(x) \& b$, for all $a, b \in Q$, $f \in Q^X$, $x \in X$. Then $(Q^X, T_X)$ is the free double
quantale module generated by $X$, equipped with the map $\varphi : x \in X \mapsto \varphi_x \in Q^X$, where $\varphi_x$ is defined by $\varphi_x(y) = \begin{cases} 0, & y \neq x, \\ e, & y = x. \end{cases}$ for all $y \in X$.

**Proof.** It’s easy to prove that $(Q^X, T_X)$ is a double quantale module. Let $(M, T_M)$ be any double quantale module and $g : X \rightarrow M$ be an arbitrary map. First observe that for all $f \in Q^X$, $Q$ be a unital quantale with unit $e$, hence $f = T_X(e, f, e)$ by definition 2.2. So every elements of $Q^X$ could denote by $T_X(c, f, d)$ for some $c, d \in Q, f \in Q^X$. Define map $h_g : Q^X \rightarrow M$ such that $h_g(T_X(c, f, d)) = \bigvee_{x \in X} T_M(c, T_M(f(x), g(x), f(x)), d)$, for all $T_X(c, f, d) \in Q^X, c, d \in Q$.

For all $x' \in Z, (h_g \circ \varphi)(x') = h_g(\varphi_{x'}(x')) = \bigvee_{x \in X} T_M(\varphi_{x'}(x), g(x), \varphi_{x'}(x)) = T_M(e, g(x), e) = f(x)$, hence $h_g \circ \varphi = f$. This implies that the following diagram commute.

![Diagram](image)

We will prove that $h_g$ is a Q-P quantale module homomorphism. For all $\{f_i\}_{i \in I}, a, b \in Q, f \in Q^X$, we have

(i) $h_g(\bigvee_{i \in I} f_i) = h_g(T_X(e, \bigvee_{i \in I} f_i), e)$

$$= \bigvee_{x \in X} T_M(e, T_M(\bigvee_{i \in I} f_i, g(x), \bigvee_{i \in I} f_i), e)$$

$$= \bigvee_{x \in X} T_M(\bigvee_{i \in I} f_i, g(x), \bigvee_{i \in I} f_i)$$

$$= \bigvee_{i \in I} \bigvee_{x \in X} T_M(f_i, g(x), f_i)$$

$$= \bigvee_{i \in I} h_g(f_i);$$

(ii) $h_g(T_X(a, f, b)) = \bigvee_{x \in X} T_M(a, T_M(f(x), g(x), f(x)), b)$

$$= T_M(a, \bigvee_{x \in X} T_M(f(x), g(x), f(x)), b)$$

$$= T_M(a, h_g(f), b).$$

Therefore, $h_g$ is a Q-P quantale module homomorphism.

Next, we will prove that $h_g$ is an unique Q-P quantale module homomorphism satisfying the above conditions.

Now, let $h'_g : Q^X \rightarrow M$ be another unique Q-P quantale module homomor-
phism such that \( h'_g \circ \varphi = g \). For all \( T_X(c, f, d) \in Q^X \), we have
\[
\begin{align*}
    h_g(T_X(c, f, d)) &= \bigvee_{x \in X} T_M(c, T_M(f(x), g(x), f(x)), d) \\
    &= \bigvee_{x \in X} T_M(c, T_M(f(x), (h'_g \circ \varphi)(x), f(x)), d) \\
    &= T_M(c, h'_g(\bigvee_{x \in X} T_X(f(x), \varphi_x, f(x))), d) \\
    &= T_M(c, h'_g(f), d) \quad (\bigvee_{x \in X} T_X(f(x), \varphi_x, f(x)) = f) \\
    &= h'_g(T_X(c, f, d)).
\end{align*}
\]

Therefore, \((Q^X, T_X)\) is the free Q-P quantale module generated by \( X \), equipped with the map \( \varphi \).

**Definition 3.9.** Let \( X \) be a nonempty set, \( Q, P \) is unital quantale , \((Q^X, T_X)\) is called **free Q-P quantale module** generated by \( X \).

**Theorem 3.10.** The forgetful functor \( U : \mathbf{QMod}_P \rightarrow \mathbf{Set} \) have a left adjoint.

**Proof.** Let \( X \) and \( Y \) be nonempty sets, \((Q^X, T_X)\) and \((Q^Y, T_Y)\) be the free Q-p quantale module generated by \( X \) and \( Y \) respectively.

Corresponding map \( f : X \rightarrow Y \) defines \( M(f) : Q^X \rightarrow Q^Y \) such that
\[
    M(f)(g)(y) = \bigvee \{g(x) \mid f(x) = y, x \in X\}, \text{ for all } g \in Q^X, y \in Y. \quad \text{Obiviously,}
\]

\( M(f) \) is well defined.

We check \( M(f) \) is a Q-P quantale module homomorphism.

For all \( g_i, g \in Q^X, a \in Q, b \in P, y \in Y \) we have
\[
\begin{align*}
    (i)M(f)(\bigvee_{i \in I} g_i) &= \bigvee_{i \in I} \{g_i(x) \mid f(x) = y, x \in X\} \\
    &= \bigvee_{i \in I} \{g_i(x) \mid f(x) = y, x \in X\} \\
    &= \bigvee_{i \in I} M(f)(g_i)(y).
\end{align*}
\]

Thus \( M(f) \) preserves arbitrary joins.
\[
\begin{align*}
    (ii)M(f)(T_X(a, g, b))(y) &= \bigvee \{T_X(a, g, b)(x) \mid f(x) = y, x \in X\} \\
    &= \bigvee \{a \& g(x) \& b \mid f(x) = y, x \in X\} \\
    &= a \& (\bigvee \{g(x) \mid f(x) = y, x \in X\}) \& b \\
    &= a \& (M(f)(g))(y) \& b \\
    &= T_Y(a, M(f)(g), b)(y).
\end{align*}
\]

Thus \( M(f)(T_X(a, g, b))(y) = T_Y(a, M(f)(g), b)(y) \). It is readily verified that \( M(f) \) is a Q-P quantale module homomorphism.

Next, we will check that \( M : \mathbf{Set} \rightarrow \mathbf{QMod}_P \) is a functor.

Let \( f : X \rightarrow Y, \ g : Y \rightarrow Z \), \( id_X \) is the identity function on \( X \). For all \( h \in Q^X, \ x \in X, \ z \in Z \), we have
(i) $M(id_X)(h)(x) = \bigvee \{h(x) \mid id_X(x) = x\} = h(x) = id_{QX}(h)(x)$, it shows that $M$ preserves identity function.

(ii) $(M(g) \circ M(f))(h)(z) = \bigvee \{M(f)(h)(y) \mid g(y) = z, y \in Y\}
= \bigvee \{\bigvee \{h(x) \mid f(x) = y, x \in X\} \mid g(y) = z, y \in Y\}
= \bigvee \{h(x) \mid f(x) = y, g(y) = z, x \in X, y \in Y\}
= \bigvee \{h(x) \mid g(f(x)) = z, x \in X\}
= M(g \circ f)(h)(z),$

then $M$ preserves composition.

Finally, we will prove that $M$ is the left adjoint of $U$.

By theorem 3.8, we have $(Q^X, T_X)$ is the free Q-P quantale module generated by $X$, equipped with the map $\varphi$, therefore, $M$ is the left adjoint of $U$.

\[ \square \]

**Theorem 3.11.** The forgetful functor $U : \textbf{qMod}_P \rightarrow \textbf{Set}$ preserves and reflects regular epimorphisms.

*Proof.* It is easy to be verified that the forgetful functor $U$ preserves regular epimorphisms. We will check the forgetful functor $U$ reflects regular epimorphisms.

At first, every regular epimorphism is a surjective homomorphism in $\textbf{qMod}_P$ by Theorem 3.7.

Next, we prove that every surjective homomorphism is a regular epimorphisms in $\textbf{qMod}_P$.

Let $h : M_1 \rightarrow M_2$ be a surjective Q-P quantale module homomorphism. Since the surjective morphism is an regular epimorphism in $\textbf{Set}$. Then $h$ is a regular epimorphism in $\textbf{Set}$, there exists a set $X$ and maps $f, g$ such that $(h, M_2)$ is a coequalizer of $f$ and $g$.

Let $(Q^X, T_X)$ be a Q-P quantale module generated by $X$. Since $Q$ be a unital quantale with unit $e$, hence $s = T_X(e,s,e)$ for all $s \in Q^X$.

Define map $h_f, h_g : Q^X \rightarrow M$ such that

$$h_f(T_X(a,s,b)) = \bigvee_{x \in X} T_{M_1}(a, T_{M_1}(s(x), f(x), s(x)), b),$$

$$h_g(T_X(a,s,b)) = \bigvee_{x \in X} T_{M_1}(a, T_{M_1}(s(x), g(x), s(x)), b),$$

for all $T_X(a,s,b) \in Q^X$, $s \in Q^X$, $a,b \in Q$.

We know that $h_f$ and $h_g$ are Q-P quantale module homomorphisms by theorem 3.8.

Since $h_f$ is a Q-P quantale module homomorphism, and $h \circ f = h \circ g$, then $h \circ h_f = h \circ h_g$. Suppose there is a Q-P quantale module homomorphism $h' : M_1 \rightarrow M_2$ with $h' \circ h_f = h' \circ h_g$, then we have $h' \circ f = h' \circ g$.

Because $(h, M_2)$ is the coequalizer of $f$ and $g$, there is a unique Q-P quantale module homomorphism $\overline{h} : M_2 \rightarrow M_3$ such that $h' = \overline{h} \circ h$. Since $h$ is a surjective of Q-P quantale module homomorphism, then there exists $x', y' \in M_1$ and $\{x'_i\}_{i \in I} \subseteq M_1$ such that $h(x_1) = x, h(y_1) = y, h(x'_i) = x_i$.  

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We check that \( \tilde{h} \) be a Q-P quantale module homomorphism in the following.

(i) \( \bigvee_{i \in I} x_i = \tilde{h}(\bigvee_{i \in I} h(x'_i)) = \tilde{h}(\bigvee_{i \in I} x'_i) = h'(\bigvee_{i \in I} x'_i) = \bigvee_{i \in I} \tilde{h}(x'_i) = \bigvee_{i \in I} h(x_i) \),

(ii) For any \( a \in Q, b \in P, m \in M_2 \), since \( h \) is a surjective of double quantale module homomorphism, there exists \( m' \in M_2 \) such that \( h(m') = m \).

So we have \( T_3(a, h(m), b) = T_3(a, h(m'), b) = h'(T_1(a, m', b)) = \tilde{h}(T_1(a, m', b)) = \tilde{h}(T_2(a, h(m'), b) = \tilde{h}(T_2(a, m, b)). \)

Hence, \((h, M_2)\) is an coequalizer of \( h_f \) and \( h_g \) in \( \text{qMod}_P \), so \( h \) is a regular epimorphism in \( \text{qMod}_P \). Therefore, the regular epimorphisms are precisely surjective homomorphisms in \( \text{qMod}_P \). Since the forgetfull functor \( U : \text{qMod}_P \to \text{Set} \) reflects surjective homomorphisms, hence \( U : \text{qMod}_P \to \text{Set} \) reflects regular epimorphisms.

The combination of theorem 3.7, theorem 3.10 and theorem 3.11, we can obtain the main result of this paper.

**Theorem 3.12.** The category \( \text{qMod}_P \) is algebraic.

**REFERENCES**


