Comparison Theorem for Oscillation of Nonlinear Delay Partial Difference Equations

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Abstract: In this paper, we consider certain nonlinear partial difference equations

\[(aA_{m+1,n} + bA_{m,n+1} + cA_{m,n})^k - (dA_{m,n})^k + \sum_{i=1}^{u} p_i(m,n)A_{m-\sigma_i,n-\tau_i}^k = 0\]

where \(a, b, c, d \in (0, \infty), d > c, k = q/p, p, q\) are positive odd integers, \(u\) is a positive integer, \(p_i(m,n), (i = 0, 1, 2, \ldots, u)\) are positive real sequences. \(\sigma_i, \tau_i \in \mathbb{N}_0 = \{1, 2, \ldots\}, i = 1, 2, \ldots, u\). A new comparison theorem for oscillation of the above equation is obtained.

Key words: Nonlinear partial difference equations; Comparison theorem; Eventually positive solutions


1. INTRODUCTION

In this paper we consider nonlinear partial difference equation

\[(aA_{m+1,n} + bA_{m,n+1} + cA_{m,n})^k - (dA_{m,n})^k + \sum_{i=1}^{u} p_i(m,n)A_{m-\sigma_i,n-\tau_i}^k = 0, \quad (1.1)\]
where \( a, b, c, d \in (0, \infty), \) \( d > c \), \( k = q/p \), \( p, q \) are positive odd integers, \( u \) is a positive integer, \( p_i(m, n), (i = 0, 1, 2, \cdots u) \) are positive real sequences. \( \sigma_i, \tau_i \in N_0 = \{1, 2, \cdots \}, i = 1, 2, \cdots , u \). The purpose of this paper is to obtain a new comparison theorem for oscillation of all solutions of (1.1).

2. MAIN RESULTS

To prove our main result we need several preparatory results.

**Lemma 2.1** Assume that \( \{A_{m,n}\} \) is a positive solution of (1.1). Then

\[
I: \quad A_{m+1,n} \leq \theta_1 A_{m,n}, A_{m,n+1} \leq \theta_2 A_{m,n}, \tag{2.1}
\]

and

\[
II: \quad A_{m-\sigma_i,n-\tau_i} \geq \theta_1^{\sigma_i} \theta_2^{\tau_i} A_{m-\sigma_0,n-\tau_0}, \tag{2.2}
\]

where \( \theta_1 = \frac{d - c}{a}, \theta_2 = \frac{d - c}{b}, \sigma_0 = \min_{1 \leq i \leq u} \{\sigma_i\}, \tau_0 = \min_{1 \leq i \leq u} \{\tau_i\} \).

**Proof.** Assume that \( \{A_{m,n}\} \) is eventually positive solutions of (1.1). From (1.1), we have

\[
(aA_{m+1,n} + bA_{m,n+1} + cA_{m,n})^k - (dA_{m,n})^k = - \sum_{i=1}^{u} p_i(m, n) A_{m-\sigma_i,n-\tau_i}^k \leq 0,
\]

and so

\[
(aA_{m+1,n} + bA_{m,n+1} + cA_{m,n})^k \leq (dA_{m,n})^k.
\]

Since \( k = \frac{p}{q} \), \( p, q \) are positive odd integers, then

\[
aA_{m+1,n} + bA_{m,n+1} \leq (d - c)A_{m,n}.
\]

Hence \( A_{m+1,n} \leq \theta_1 A_{m,n} \) and \( A_{m,n+1} \leq \theta_2 A_{m,n} \). From the above inequality, we can find \( A_{m,n} \leq \theta_1^{\sigma_0} A_{m-\sigma_0,n-\tau_0} \leq \theta_1^{\sigma_0} \theta_2^{\tau_0} A_{m-\sigma_0,n-\tau_0} \), and

\[
A_{m-\sigma_i,n-\tau_i} \leq \theta_1^{\sigma_i} \theta_2^{\tau_i} A_{m-\sigma_0,n-\tau_0} \leq \theta_2^{\tau_i} A_{m-\sigma_i,n-\tau_i}.
\]

Hence

\[
A_{m,n} \leq \theta_1^{\sigma_0} \theta_2^{\tau_0} A_{m-\sigma_0,n-\tau_0} \leq \theta_1^{\sigma_i} \theta_2^{\tau_i} A_{m-\sigma_i,n-\tau_i}.
\]

The proof of Lemma 2.1 is completed. \( \square \)

**Lemma 2.2** [1] If \( x, y \in R^+ \) and \( x \neq y \), then

\[
rx^{r-1}(x - y) > x^r - y^r > ry^{r-1}(x - y), \quad \text{for} \quad r > 1.
\]

**Theorem 2.1** If the difference inequality

\[
aA_{m+1,n} + bA_{m,n+1} -(d-c)A_{m,n} + \sum_{i=1}^{u} \frac{\theta_1^{\sigma_i} \theta_2^{\tau_i}}{kd^{k-1}} p_i(m, n) A_{m-\sigma_i,n-\tau_i} \leq 0 \tag{2.3}
\]

has no eventually positive solutions, then every solution of Equation (1.1) oscillates.
**Comparison Theorem for Oscillation of Nonlinear Delay Partial Difference Equations**

**Proof.** Assume that \(\{A_{m,n}\}\) is a positive solution of Equation (1.1). Then, by (1.1) and Lemma 2.2, we obtain

\[
aA_{m+1,n} + bA_{m,n+1} - (d - c)A_{m,n} + \sum_{i=1}^{u} p_i(m, n) \frac{A_{m-i,n-i-1}}{kd^k-1} A_{m,n}^k 
\leq 0 \quad (2.4)
\]

Substituting (2.2) into (2.4), we have

\[
aA_{m+1,n} + bA_{m,n+1} - (d - c)A_{m,n} + \sum_{i=1}^{u} \left( \frac{\theta_1^{\sigma_0 - k\sigma_i} \theta_2^{\tau_0 - k\tau_i}}{kd^k-1} \right) p_i(m, n) A_{m-i,n-i-1} \leq 0.
\]

This contradiction completes the proof. \(\square\)

Define a set \(E\) by

\[
E = \{ \lambda > 0 \mid d - c - \lambda Q_{m,n} > 0, \quad \text{eventually} \}
\]

where \(Q_{m,n} = \sum_{i=1}^{u} \frac{\theta_1^{\sigma_0 - k\sigma_i} \theta_2^{\tau_0 - k\tau_i}}{kd^k-1} p_i(m, n).\)

**Theorem 2.2** Assume that

(i) \(\lim_{m,n \to \infty} \sup Q_{m,n} > 0;\)

(ii) there exists \(M \geq m_0, N \geq n_0\) such that if \(\sigma_0 > \tau_0 > 0,\)

\[
\sup_{\lambda \in E,M \geq m_0,N \geq n_0} \lambda^{\sigma_0 - \tau_0} \left[ \prod_{j=1}^{\sigma_0} \prod_{i=1}^{\tau_0} \left( (d - c - \lambda Q_{m-i,j,n-1}) \right) \right] < \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\tau_0} (\theta_1^{\sigma_0 - \sigma_0}, \quad (2.5)
\]

and if \(\tau_0 > \sigma_0 > 0,\)

\[
\sup_{\lambda \in E,M \geq m_0,N \geq n_0} \lambda^{\sigma_0 - \tau_0} \left[ \prod_{j=1}^{\sigma_0} \prod_{i=1}^{\tau_0} \left( (d - c - \lambda Q_{m-i,j,n-1}) \right) \right] < \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\tau_0} (\theta_1^{\sigma_0 - \tau_0}). \quad (2.6)
\]

Then every solution of (1.1) oscillates.

**Proof.** Suppose, to the contrary, \(A_{m,n}\) is an eventually positive solution. We define a subset \(S\) of the positive numbers as follows:

\[S(\lambda) = \{ \lambda > 0 \mid aA_{m+1,n} + bA_{m,n+1} - [(d - c - \lambda Q_{m,n}) A_{m,n} \leq 0, \quad \text{eventually}\}\}.

From (2.3) and Lemma 2.1, we have

\[
aA_{m+1,n} + bA_{m,n+1} - (d - c - \theta_1^{\sigma_0} \theta_2^{\tau_0} Q_{m,n}) A_{m,n} \leq 0,
\]

which implies \(\theta_1^{\sigma_0} \theta_2^{\tau_0} \in S(\lambda)\). Hence, \(S(\lambda)\) is nonempty. For \(\lambda \in S\), we have eventually that \(d - c - \lambda Q_{m,n} > 0\), which implies that \(S \subseteq E\). Due to condition (i), the set \(E\) is bounded, and hence, \(S(\lambda)\) is bounded. Let \(u \in S\). Then from Lemma 2.1, we have

\[
\left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right) A_{m+1,n+1} \leq aA_{m+1,n} + bA_{m,n+1} 
\leq (d - c - uQ_{m,n}) A_{m,n}.
\]

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If $\sigma_0 > \tau_0 > 0$, then

$$A_{m,n} \leq \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{-\tau_0} \prod_{i=1}^{\tau_0} (d - c - uQ_{m-i,n-i}) A_{m-\tau_0,n-\tau_0};$$

and for $j = 1, 2, \ldots, \sigma_0 - \tau_0$, we have

$$A_{m-j,n} \leq \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{-\tau_0} \prod_{i=1}^{\tau_0} (d - c - uQ_{m-i-j,n-i}) A_{m-\tau_0-j,n-\tau_0} \leq \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{-\tau_0} \prod_{i=1}^{\tau_0} (d - c - uQ_{m-i-j,n-i}) A_{m-\sigma_0,n-\tau_0}. \tag{2.7}$$

Now, from Lemma 2.1 and (2.7), it follows that

$$A^{\sigma_0-\tau_0} \leq \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{-\tau_0} \theta_1^{\sigma_0-\tau_0} \theta_0 \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{-\tau_0} \prod_{i=1}^{\tau_0} (d - c - uQ_{m-i-j,n-i}) A_{m-\sigma_0,n-\tau_0};$$

i. e.,

$$A_{m,n} \leq \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{-\sigma_0} \theta_1^{\sigma_0-\sigma_0} \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{-\sigma_0} \prod_{i=1}^{\tau_0} (d - c - uQ_{m-i,n-i}) \frac{1}{\tau_0-\sigma_0} A_{m-\sigma_0,n-\tau_0}. \tag{2.8}$$

Similarly, if $\tau_0 > \sigma_0 > 0$, then

$$A_{m,n} \leq \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{-\sigma_0} \theta_1^{\sigma_0-\sigma_0} \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{-\sigma_0} \prod_{i=1}^{\tau_0} (d - c - uQ_{m-i,n-i}) \frac{1}{\tau_0-\sigma_0} A_{m-\sigma_0,n-\tau_0}. \tag{2.9}$$

Substituting (2.8) and (2.9) into (2.3), we get respectively, for $\sigma_0 > \tau_0$,

$$aA_{m+1,n} + bA_{m,n+1} - (d-c)A_{m,n}$$

$$+ Q_{m,n} \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\sigma_0-\sigma_0} \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\sigma_0-\sigma_0} \prod_{i=1}^{\tau_0} (d - c - uQ_{m-i,n-i}) \frac{1}{\tau_0-\sigma_0} A_{m,n} \leq 0,$$

and for $\tau_0 > \sigma_0$,

$$aA_{m+1,n} + bA_{m,n+1} - (d-c)A_{m,n}$$

$$+ Q_{m,n} \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\sigma_0-\sigma_0} \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\sigma_0-\sigma_0} \prod_{i=1}^{\tau_0} (d - c - uQ_{m-i,n-i}) \frac{1}{\tau_0-\sigma_0} A_{m,n} \leq 0.$$

Hence, for $\sigma_0 > \tau_0$,

$$aA_{m+1,n} + bA_{m,n+1} - \left\{ d - c - Q_{m,n} \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\sigma_0-\sigma_0} \right\} \times \sup_{m \geq M, n \geq N} \left[ \prod_{i=1}^{\tau_0} (d - c - uQ_{m-i-j,n-i}) \frac{1}{\tau_0-\sigma_0} \right] A_{m,n} \leq 0. \tag{2.10}$$
and for $\tau_0 > \sigma_0$,
\[
\begin{aligned}
aA_{m+1,n} + bA_{m,n+1} - \{d - c - Q_{m,n}\frac{a}{\theta_2} + \frac{b}{\theta_1}\sigma_0\sigma_0 - \tau_0}
\times \sup_{m \geq M, n \geq N} \left[ \prod_{j=1}^{\tau_0 - \sigma_0} \prod_{i=1}^{\sigma_0} (d - c - uQ_{m-i,n-i-j}) \right]^{\frac{1}{\sigma_0 - \tau_0}} A_{m,n} \leq 0.
\end{aligned}
\tag{2.11}
\]

From (2.10) and (2.11), we get
\[
\begin{aligned}
\left(\frac{a}{\theta_2} + \frac{b}{\theta_1}\right)^{\sigma_0 - \tau_0} \left\{ \sup_{m \geq M, n \geq N} \left[ \prod_{j=1}^{\tau_0 - \sigma_0} \prod_{i=1}^{\sigma_0} (d - c - uQ_{m-i,n-i-j}) \right]^{\frac{1}{\sigma_0 - \tau_0}} \right\} \in S \text{ for } \sigma_0 > \tau_0,
\end{aligned}
\tag{2.12}
\]
and
\[
\begin{aligned}
\left(\frac{a}{\theta_2} + \frac{b}{\theta_1}\right)^{\tau_0 - \sigma_0} \left(\sup_{m \geq M, n \geq N} \left[ \prod_{j=1}^{\tau_0 - \sigma_0} \prod_{i=1}^{\sigma_0} (d - c - uQ_{m-i,n-i-j}) \right]^{\frac{1}{\tau_0 - \tau_0}} \right) \in S \text{ for } \tau_0 > \sigma_0.
\end{aligned}
\tag{2.13}
\]

On the other hand, (2.5) implies that there exists $a_1 \in (0,1)$ (we can choose the same) such that for $\sigma_0 > \tau_0$
\[
\begin{aligned}
\sup_{\lambda \in E, M \geq M, n \geq N} \lambda \left[ \prod_{j=1}^{\tau_0 - \sigma_0} \prod_{i=1}^{\sigma_0} (d - c - \lambda Q_{m-i,n-i-j}) \right]^{\frac{1}{\tau_0 - \tau_0}} \leq a_1 \left(\frac{a}{\theta_2} + \frac{b}{\theta_1}\right)^{\tau_0 - \sigma_0}, \tag{2.14}
\end{aligned}
\]
and (2.6) implies that there exists $a_1 \in (0,1)$ (we can choose the same) such that for $\tau_0 > \sigma_0 > 0$,
\[
\begin{aligned}
\sup_{\lambda \in E, M \geq M, n \geq n} \lambda \left[ \prod_{j=1}^{\tau_0 - \sigma_0} \prod_{i=1}^{\sigma_0} (d - c - \lambda Q_{m-i,n-i-j}) \right]^{\frac{1}{\tau_0 - \tau_0}} \leq a_1 \left(\frac{a}{\theta_2} + \frac{b}{\theta_1}\right)^{\tau_0 - \sigma_0}. \tag{2.15}
\end{aligned}
\]

In particular, (2.14) and (2.15) lead to (when $\lambda = u$), respectively,
\[
\begin{aligned}
\left(\frac{a}{\theta_2} + \frac{b}{\theta_1}\right)^{\tau_0 - \sigma_0} \sup_{\lambda \in E, M \geq M, n \geq N} \left[ \prod_{j=1}^{\tau_0 - \sigma_0} \prod_{i=1}^{\sigma_0} (d - c - uQ_{m-i,n-i-j}) \right]^{\frac{1}{\tau_0 - \tau_0}} \geq \frac{u}{a_1} \text{ for } \sigma_0 > \tau_0, \tag{2.16}
\end{aligned}
\]
and
\[
\begin{aligned}
\left(\frac{a}{\theta_2} + \frac{b}{\theta_1}\right)^{\tau_0 - \sigma_0} \sup_{\lambda \in E, M \geq M, n \geq N} \left[ \prod_{j=1}^{\tau_0 - \sigma_0} \prod_{i=1}^{\sigma_0} (d - c - uQ_{m-i,n-i-j}) \right]^{\frac{1}{\tau_0 - \tau_0}} \geq \frac{u}{a_1} \text{ for } \tau_0 > \sigma_0. \tag{2.17}
\end{aligned}
\]

Since $u \in S$ and $u' \leq u$ implies that $u' \in S$, it follows from (2.12) and (2.16) for $\sigma_0 > \tau_0$, (2.13) and (2.17) for $\tau_0 > \sigma_0$ that $\frac{u}{a_1} \in S$. Repeating the above arguments with $u$ replaced by $\frac{u}{a_1}$, we get $\frac{u}{a_1a_2} \in S$, where $a_2 \in (0,1)$. Continuing in this way, we obtain $\frac{u}{\prod_{i=1}^{\infty} a_i} \in S$, where $a_i \in (0,1)$. This contradicts the boundedness of $S$. The proof is complete. \qed
Corollary 2.1 In addition to (i) of Theorem 2.1, assume that for \( \sigma_0 > \tau_0 > 0, \)
\[
\lim_{m,n \to \infty} \inf_{1} \frac{1}{(\sigma_0 - \tau_0)\tau_0} \sum_{j=1}^{\tau_0} \sum_{i=1}^{\tau_0} Q_{m-i-j,n-i} > \frac{(d-c)^{\tau_0+1} - \tau_0}{(\tau_0 + 1)^{\tau_0+1}} \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{-\tau_0} \theta_1^{\sigma_0 - \tau_0},
\]
and for \( \tau_0 > \sigma_0 > 0, \)
\[
\lim_{m,n \to \infty} \inf_{1} \frac{1}{(\sigma_0 - \tau_0)\tau_0} \sum_{j=1}^{\tau_0} \sum_{i=1}^{\tau_0} Q_{m-i-j,n-i} > \frac{(d-c)^{\sigma_0+1} - \sigma_0}{(\sigma_0 + 1)^{\sigma_0+1}} \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{-\sigma_0} \theta_2^{\sigma_0 - \tau_0}.
\]

Then every solution of (1.1) oscillates.

Proof. We note that
\[
\max_{\sigma > \lambda > 0} \lambda (d-c - \lambda e)^{\tau_0} = \frac{(d-c)^{\tau_0+1} - \tau_0}{e(\tau_0 + 1)^{\tau_0+1}}.
\]

We shall use this for
\[
e = \frac{1}{(\sigma_0 - \tau_0)\tau_0} \sum_{j=1}^{\tau_0} \sum_{i=1}^{\tau_0} Q_{m-i-j,n-i}.
\]

Clearly,
\[
\lambda \prod_{j=1}^{\tau_0} \prod_{i=1}^{\tau_0} (d-c - \lambda Q_{m-i-j,n-i}) \frac{1}{(\sigma_0 - \tau_0)}
\]
\[
\leq \lambda \left[ \frac{1}{(\sigma_0 - \tau_0)} \sum_{j=1}^{\tau_0} \sum_{i=1}^{\tau_0} (d-c - \lambda Q_{m-i-j,n-i}) \right]^{\tau_0}
\]
\[
\leq \lambda (d-c - \frac{\lambda}{(\sigma_0 - \tau_0)}) \prod_{j=1}^{\tau_0} \sum_{i=1}^{\tau_0} Q_{m-i-j,n-i} \right]^{\tau_0}
\]
\[
\leq \frac{(d-c)^{\tau_0+1} - \tau_0}{(\tau_0 + 1)^{\tau_0+1}} \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\tau_0} \theta_1^{\sigma_0 - \sigma_0}.
\]

Similarly, we have
\[
\lambda \prod_{j=1}^{\tau_0} \prod_{i=1}^{\tau_0} (d-c - \lambda Q_{m-i,j,n-i}) \frac{1}{(\sigma_0 - \tau_0)} < \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\sigma_0} \theta_2^{\sigma_0 - \tau_0}.
\]

By Theorem 2.1, every solutions of (1.1) oscillates. The proof is complete. \(\square\)

By a similar argument, we have the following results:

Theorem 2.3 Assume that
(i) \( \lim_{m,n \to \infty} \sup Q_{m,n} > 0; \)
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(ii) there exists \( M \geq m_0, N \geq n_0 \) such that if \( \sigma_0 = \tau_0 > 0 \),

\[
\sup_{\lambda \in E, M \geq m, N \geq n} \lambda \prod_{i=1}^{\sigma_0} (d - c - \lambda Q_{m-i,n-i}) < \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{\sigma_0}
\]

Then every solution of (1.1) oscillates.

**Corollary 2.2** If the condition of Theorem 2.2 holds, and

\[
\lim_{m,n \to \infty} \inf Q_{m,n} = q > (d - c)^{\sigma_0+1} \sigma_0 a \theta_2 \left( \frac{a}{\theta_2} + \frac{b}{\theta_1} \right)^{-\sigma_0},
\]

Then every solution of (1.1) oscillates.

**Theorem 2.4** Assume that

(i) \( \lim_{m,n \to \infty} \sup Q_{m,n} > 0 \);

(ii) there exists \( M \geq m_0, N \geq n_0 \) such that either

\[
\sup_{\lambda \in E, M \geq m, N \geq n} \lambda \prod_{j=1}^{\tau_0} \prod_{i=1}^{\sigma_0} (d - c - \lambda Q_{m-i,n-j}) \left( d - c - \lambda Q_{m-i,n-j} \right)^{\frac{1}{\sigma_0}} < a^{\sigma_0} \theta_2^{\tau_0},
\]

or

\[
\sup_{\lambda \in E, M \geq m, N \geq n} \lambda \prod_{i=1}^{\sigma_0} \prod_{j=1}^{\tau_0} (d - c - \lambda Q_{m-i,n-j}) \left( d - c - \lambda Q_{m-i,n-j} \right)^{\frac{1}{\sigma_0}} < b^{\tau_0} \theta_1^{\sigma_0}.
\]

Then every solution of (1.1) oscillates.

**Corollary 2.3** In addition to (i) of Theorem 2.3, assume that for \( \sigma_0, \tau_0 > 0 \), either

\[
\lim_{m,n \to \infty} \inf \frac{1}{\sigma_0 \tau_0} \sum_{j=1}^{\tau_0} \sum_{i=1}^{\sigma_0} Q_{m-i,n-j} > a^{-\sigma_0} \theta_2^{\tau_0} \frac{\sigma_0^{\tau_0}}{(\sigma_0 + 1)^{\sigma_0+1}},
\]

or

\[
\lim_{m,n \to \infty} \inf \frac{1}{\sigma_0 \tau_0} \sum_{i=1}^{\sigma_0} \sum_{j=1}^{\tau_0} Q_{m-i,n-j} > b^{-\tau_0} \theta_1^{\sigma_0} \frac{\sigma_0^{\tau_0}}{(\tau_0 + 1)^{\tau_0+1}},
\]

Then every solution of (1.1) oscillates.

**Theorem 2.5** Assume that

(i) \( \lim_{m,n \to \infty} \sup Q_{m,n} > 0 \);

(ii) there exists \( M \geq m_0, N \geq n_0 \) such that if \( \sigma_0 > 0, \tau_0 = 0 \),

\[
\sup_{\lambda \in E, M \geq m, N \geq n} \lambda \prod_{i=1}^{\sigma_0} (d - c - \lambda Q_{m-i,n}) < a^{\sigma_0}.
\]

Then every solution of (1.1) oscillates.

**Corollary 2.4** In addition to (i) of Theorem 2.4, assume that \( \sigma_0 > 0, \tau_0 = 0 \), and

\[
\lim_{m,n \to \infty} \inf Q_{m,n} > \frac{(d - c)^{\sigma_0+1} \sigma_0^{\tau_0}}{a^{\sigma_0}(\sigma_0 + 1)^{\sigma_0+1}}.
\]

Then every solution of (1.1) oscillates.

**Theorem 2.6** Assume that

(i) \( \lim_{m,n \to \infty} \sup Q_{m,n} > 0 \);
(ii) there exists $M \geq m_0$, $N \geq n_0$ such that if $\sigma_0 = 0$, $\tau_0 > 0$,
\[
\sup_{\lambda \in E, M \geq m, N \geq n} \lambda \prod_{j=1}^{\tau_0} (d - c - \lambda Q_{m,n-j}) < b^{\tau_0}.
\]

Then every solution of (1.1) oscillates.

**Corollary 2.5** In addition to (i) of Theorem 2.5, assume that $\sigma_0 = 0$, $\tau_0 > 0$, and
\[
\lim_{m,n \to \infty} \inf Q_{m,n} > \frac{(d - c)^{\tau_0 + 1}}{b^{\tau_0} (\tau_0 + 1)^{\tau_0 + 1}}.
\]

Then every solution of (1.1) oscillates.

**Theorem 2.7** Assume that
(i) $\lim_{m,n \to \infty} \sup Q_{m,n} > 0$;
(ii) for $\sigma_0, \tau_0 > 0$, $\lim_{m,n \to \infty} \inf Q_{m,n} = q > 0$, (2.18)
and
\[
\lim_{m,n \to \infty} Q_{m,n} > (d - c) \theta_1^{\sigma_0} \theta_2^{\tau_0} - \frac{a \theta_1 + b \theta_2}{(d - c)} q > 0. \tag{2.19}
\]

Then every solution of (1.1) oscillates.

**Proof.** Suppose, to the contrary, $A_{m,n}$ is an eventually positive solution. From (2.3) and (2.18), for any $\epsilon > 0$, we have $Q_{m,n} > q - \epsilon$ for $m \geq M, n \geq N$. From (2.3), Lemma 2.1 and above inequality, we obtain
\[
A_{m,n} \geq \frac{(q - \epsilon)}{(d - c)} A_{m-\sigma_0,n-\tau_0} \geq \frac{(q - \epsilon)}{(d - c)} \theta_1^{1-\sigma_0} \theta_2^{1-\tau_0} A_{m-1,n-1},
\]

$A_{m,n} \geq \frac{(q - \epsilon)}{(d - c)} \theta_1^{1-\sigma_0} \theta_2^{1-\tau_0} A_{m-1,n-1}$, and $A_{m,n} \geq \frac{(q - \epsilon)}{(d - c)} \theta_1^{1-\sigma_0} \theta_2^{1-\tau_0} A_{m,n-1}$.

Substituting above inequalities into (2.3), we get
\[
\left[ \frac{a \theta_1^{1-\sigma_0} \theta_2^{1-\tau_0} + b \theta_1^{1-\sigma_0} \theta_2^{1-\tau_0}}{d - c} (q - \epsilon) - (d - c) + Q_{m,n} \theta_1^{\sigma_0} \theta_2^{\tau_0} \right] A_{m,n} < 0,
\]
which implies
\[
\lim_{m,n \to \infty} Q_{m,n} \leq (d - c) \theta_1^{\sigma_0} \theta_2^{\tau_0} - \frac{a \theta_1 + b \theta_2}{(d - c)} q > 0.
\]
This contradicts (2.19). The proof is complete. \qed

**Theorem 2.8** Assume that
(i) $\lim_{m,n \to \infty} \sup Q_{m,n} > 0$;
(ii) $\sigma_0 = \tau_0 = 0$, and
\[
\lim_{m,n \to \infty} \sup Q_{m,n} > d - c. \tag{2.20}
\]

Then every solution of (1.1) oscillates.
Proof. Let $u \in S$. Then from (2.3) and Lemma 2.1, we have $-(d-c) + Q_{m,n} A_{m,n} < 0$, which implies $\lim_{m,n \to \infty} \sup Q_{m,n} \leq d - c$. This contradicts (2.20). The proof is complete. 

REFERENCES


