Nonoscillation for System of Neutral Delay Dynamic Equation on Time Scales

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Abstract: In this paper, by fixed theorem, some sufficient conditions for nonoscillation of the system of neutral delay dynamic equations on time scales $T$ are established. Our results as special case when $T = \mathbb{R}$ and $T = \mathbb{N}$, involve and improve some known results.

Key words: Nonoscillation; System; Dynamic equations; Time scales


1. INTRODUCTION

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph.D. thesis in 1988 in order to unify continuous and discrete analysis [1]. A time scale $T$ is an arbitrary nonempty closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications [9].

Figure 1. Some Time Scales
where $T = h\mathbb{Z} = \{hk : k \in \mathbb{Z}, h > 0\}$, $\mathbb{P}_{a,b} = \bigcup_{k=0}^{\infty} [k(a + b), k(a + b) + a]$.

On any time scale $T$, we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s > t : s \in T\}, \quad \rho(t) := \sup\{s < t : s \in T\}.$$ 

A point $t \in T$, $t > \inf T$, is said to be left-dense if $\rho(t) = t$, right-dense if $t < \sup T$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The graininess function $\mu$ for a time scale $T$ is defined by

$$\mu(t) := \sigma(t) - t.$$ 

A function $f : T \to \mathbb{R}$ is called rd-continuous function provided it is continuous at right-dense points in $T$ and its left-sided limits exist (finite) at left-dense points in $T$. The set of rd-continuous functions $f : T \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(T) = C_{rd}(T, \mathbb{R})$.

Let $f$ be a differentiable function on $[a, b]$. Then $f$ is increasing, decreasing, nondecreasing, and non-increasing on $[a, b]$, if $f^\Delta(t) > 0$, $f^\Delta(t) < 0$, $f^\Delta(t) \geq 0$, and $f^\Delta(t) \leq 0$ for all $t \in [a, b]$, respectively.

For a function $f : T \to \mathbb{R}$ (the range $\mathbb{R}$ of $f$ may be actually replaced by any Banach space) the delta derivative is defined by

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if $f$ is continuous at $t$ and $t$ is right-scattered. We will make use of the following product and quotient rules for the derivative of the product $fg$ and the quotient $\frac{f}{g}$ (where $gg^\sigma \neq 0$) of two differentiable functions $f$ and $g$

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma.$$ 

$$(\frac{f}{g})^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}.$$ 

For $t_0, b \in T$, and a differentiable function $f$, the Cauchy integral of $f^\Delta$ is defined by

$$\int_{t_0}^{b} f^\Delta(t) \Delta t = f(b) - f(t_0).$$ 

An integration by parts formula reads

$$\int_{t_0}^{b} f(t) g^\Delta(t) \Delta t = [f(t)g(t)]_{t_0}^{b} - \int_{t_0}^{b} f^\Delta(t) g^\sigma(t) \Delta t.$$ 

and infinite integral is defined as

$$\int_{t_0}^{\infty} f(t) \Delta t = \lim_{b \to \infty} \int_{t_0}^{b} f(t) \Delta t.$$ 

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various equations on time scales. We refer the reader to the papers [2–8] and the reference cited therein.

In this paper, we consider dynamic equations

$$(x(t) + px(t - \tau))^\Delta + Q(t)x(t - \tau) = 0,$$  

(1)
and

\[(x(t) + Bx(t - \tau))^{\Delta} + Q(t)x(t - \tau) = 0,\]  \tag{2}

Where \(t \in \mathbb{T}, p \in \mathbb{R}, 0 \leq \tau, \sigma \in \mathbb{T}\), and \(Q(t)\) is \(rd\)-continued \(n \times n\) matrix on \([t_0, \infty) \cap \mathbb{T}\). Let \(x(t)\) be the set of all \(rd\)-continuous and bounded \(n\)-dimension vector functions on \([t_0, \infty) \cap \mathbb{T}\), and \(B\) is a \(n \times n\) matrix, and \(\|B\| \neq 0\). Let \(\|B\| = b\).

2. NONOSCILLATION THEOREMS

First we consider the case \(p\) is a constant.

**Theorem 2.1** Suppose that \(\int_{t_1}^{\infty} \|Q(s)\| \Delta s < \infty\), where \(p \neq -1\), and \(\|\cdot\|\) is any norm in \(\mathbb{T}\), then Eq.(1) has a nonoscillatory solution.

**Proof.** The proof of this theorem will be divided into five cases depending on the five different ranges of the parameter \(p\).

**Case 1.** \(0 \leq p < 1\).

Choose a \(t_1 \in \mathbb{T}\) sufficiently large such that \(t_1 \geq t_0 + \sigma\), where \(\sigma = \max\{\tau, \sigma\}\), and

\(\int_{t_1}^{\infty} \|Q(s)\| \Delta s \leq \frac{1 - p(1 + M_2) - M_1}{M_2}\)

holds. Where \(0 < M_1 < 1, M_2 > M_1, M_1 + M_2 < 2\), and \(1 - \frac{M_1 + M_2}{2} \leq p \leq \frac{1 - M_1}{1 + M_2}\).

Let \(x(t)\) be the set of all continuous and bounded vector functions on \([t_0, \infty) \cap \mathbb{T}\). Let \(A = \{x \in X : M_1 \leq \|x(t)\| \leq M_2, t_0 \leq t\}\). Define a mapping \(F : A \rightarrow X\) as follows

\[(Fx)(t) = \begin{cases} 
(1 - p)e - px(t - \tau) + \int_{t}^{\infty} Q(s)x(s - \sigma)\Delta s, & t \geq t_1; \\
(Fx)(t_1), & t_0 \leq t \leq t_1.
\end{cases} \tag{3}

Clearly \(Fx\) is \(rd\)-continuous. For every \(x \in A\) and \(t \geq t_1\), using Eq.(3), we get

\[
\|(Fx)(t)\| = \|(1 - p)e - px(t - \tau) + \int_{t}^{\infty} Q(s)x(s - \sigma)\Delta s\|
\leq \|(1 - p)e\| + \|px(t - \tau)\| + \int_{t}^{\infty} \|Q(s)x(s - \sigma)\| \Delta s
\leq (1 - p) + p\|x(t - \tau)\| + \int_{t}^{\infty} \|Q(s)x(s - \sigma)\| \Delta s
\leq 1 - p + pM_2 + M_2 \int_{t}^{\infty} \|Q(s)\| \Delta s
\leq 1 - p + pM_2 + M_2 \frac{1 - p(1 + M_2) - M_1}{M_2}
= 2(1 - p) - M_1 \leq M_2.
\]

Further, in view of Eq.(3), we have

\[
\|(Fx)(t)\| = \|(1 - p)e - px(t - \tau) - \int_{t}^{\infty} Q(s)x(s - \sigma)\Delta s\|
\geq \|(1 - p)e\| - \|px(t - \tau) - \int_{t}^{\infty} Q(s)x(s - \sigma)\Delta s\|
\]
\[\begin{align*}
&\geq 1 - p - p\|x(t - \tau)\| - \int_t^\infty \|Q(s)x(s - \sigma)\|\Delta s \\
&\geq 1 - p - pM_2 - M_2\int_t^\infty \|Q(s)\|\Delta s \\
&\geq 1 - p - pM_2 - M_2 \frac{1 - p(1 + M_2) - M_1}{M_2} \\
&= M_1.
\end{align*}\]

Thus we proved that \(FA \subseteq A\).

Now, for all \(x_1, x_2 \in A\), and \(t \geq t_1\), we have
\[
\|(Fx_1)(t) - (Fx_2)(t)\| \\
= \| - B[x_1(t - \tau) - x_2(t - \tau)] + \int_t^\infty Q(s)[x_1(s - \sigma) - x_2(s - \sigma)]\Delta s\| \\
\leq \| - B[x_1(t - \tau) - x_2(t - \tau)]\| + \|\int_t^\infty Q(s)[x_1(s - \sigma) - x_2(s - \sigma)]\Delta s\| \\
\leq b\|x_1 - x_2\| + \|x_1 - x_2\|\int_t^\infty \|Q(s)\|\Delta s \\
\leq r_1\|x_1 - x_2\| \ (r_1 = b + \int_t^\infty \|Q(s)\|\Delta s)
\]

Clearly,
\[
q_1 = p + \int_t^\infty \|Q(s)\|\Delta s \leq p + \frac{1 - p(1 + M_2) - M_1}{M_2} = \frac{1 - p - M_1}{M_2} < 1.
\]

so
\[
\|(Fx_1)(t) - (Fx_2)(t)\| \leq q_1\|x_1 - x_2\|,
\]
which proves that \(F\) is a contraction mapping. Consequently, \(F\) has the fixed point \(x\) with \(Fx = x\), \(\|x\| > 0\), for all \(t \geq t_1\), which is a nonoscillatory solution of Eq.(1).

**Case 2.** \(1 < p < \infty\).
Choose a \(t_1 \in \mathbb{T}\) sufficiently large such that \(t_1 + \tau \geq t_0 + \sigma\), and
\[
\int_{t_1 + \tau}^\infty \|Q(s)\|\Delta s \leq \frac{p - 1 - pn_1 - n_2}{n_2},
\]
hold, where \(0 < n_1 < 1\), \(n_2 > n_1\) and \(n_1 + n_2 < 2\), \(\frac{1 + n_2}{1 - n_1} < p \leq \frac{2}{2 - n_1 - n_2}\).

Let \(X(t)\) be the set of all rd-continuous and bounded vector functions on \(t_0 \leq t \in \mathbb{T}\). Set \(A = \{x \in X : n_1 \leq \|x(t)\| \leq n_2, \ t_0 \leq t \in \mathbb{T}\}\). Define a mapping \(F : A \rightarrow X\) as follows
\[
(Fx)(t) = \begin{cases} 
(1 - \frac{1}{p})e - \frac{1}{p}x(t + \tau) + \frac{1}{p} \int_t^\infty Q(s)x(s - \sigma)\Delta s, & t \geq t_1; \\
(Fx)(t_1), & t_0 \leq t \leq t_1.
\end{cases} \tag{4}
\]

Clearly \(Fx\) is \(C_{rd}\) continuous. For every \(x \in A\) and \(t \geq t_1\), using (4) we get
\[
\|(Fx)(t)\| = \|(1 - \frac{1}{p})e - \frac{1}{p}x(t + \tau) + \frac{1}{p} \int_t^\infty Q(s)x(s - \sigma)\Delta s\| \\
\]
Now, for all \( x_1, x_2 \in A \), and \( t \geq t_1 \), we have

\[
\|(Fx_1(t)) - (Fx_2(t))\| = \| - B^{-1} [x_1(t - \tau) - x_2(t - \tau)] + B^{-1} \int_{t}^{\infty} Q(s) [x_1(s - \sigma) - x_2(s - \sigma)] \Delta s \|
\]

\[
\leq \| - B^{-1} [x_1(t - \tau) - x_2(t - \tau)] \| + B^{-1} \| \int_{t}^{\infty} Q(s) [x_1(s - \sigma) - x_2(s - \sigma)] \Delta s \|
\]

\[
\leq \frac{1}{b} \| x_1(t - \tau) - x_2(t - \tau) \| + \frac{1}{b} \| \int_{t}^{\infty} Q(s) \Delta s \|
\]

\[
\leq r_2 \| x_1 - x_2 \| \quad (r_2 = \frac{1}{b} + \frac{1}{b} \int_{t}^{\infty} \| Q(s) \Delta s \|).
\]

Clearly,

\[
q_2 = \frac{1}{p} + \frac{1}{p} \int_{t}^{\infty} \| Q(s) \Delta s \| \leq \frac{1}{p} + \frac{p - 1 - pN_1 - N_2}{N_2} = \frac{p - 1 - pN_1}{pN_2} < 1.
\]
so

\[ \| (Fx_1)(t) - (Fx_2)(t) \| \leq q_2 \| x_1 - x_2 \|, \]

which proves that \( F \) is a contraction mapping. Consequently, \( F \) has the fixed point \( x \) with \( Fx = x \), \( \| x \| > 0 \), for all \( t \geq t_1 \), which is a nonoscillatory solution of Eq. (1).

**Case 3.** \( p = 1 \).

Choose a \( t_1 \in T \) sufficiently large such that \( t_1 + \tau \geq t_0 + \sigma \),

\[
\sum_{i=0}^{\infty} \int_{t_1 + (2i-1)\tau}^{t_1 + 2i\tau} \| Q(s) \| \Delta s \leq \frac{\| P \| - p_1}{p_2},
\]

where \( P \) be a nonzero constant vector and \( p_1 < p_2 \) are positive constants such that,

\[
p_1 < \| P \| \leq \frac{p_1 + p_2}{2}.
\]

Let \( X(t) \) be the set of all \( rd \)--continuous and bounded vector functions on \( t_0 \leq t \in T \). Set \( A = \{ x \in X : p_1 \leq \| x(t) \| \leq p_2, t_0 \leq t \} \). Define a mapping \( F : A \rightarrow X \) as follows

\[
(Fx)(t) = \begin{cases} 
P + \sum_{i=0}^{\infty} \int_{t_1 + (2i-1)\tau}^{t_1 + 2i\tau} Q(s)x(s-\sigma)\Delta s, & t \geq t_1; \\
(Fx)(t_1), & t_0 \leq t \leq t_1.
\end{cases}
\]

Clearly \( Fx \) is \( rd \)--continuous. For every \( x \in A \) and \( t \geq t_1 \), we have

\[
\| (Fx)(t) \| = \| P + \sum_{i=0}^{\infty} \int_{t_1 + (2i-1)\tau}^{t_1 + 2i\tau} Q(s)x(s-\sigma)\Delta s \|
\leq \| P \| + \sum_{i=0}^{\infty} \int_{t_1 + (2i-1)\tau}^{t_1 + 2i\tau} \| Q(s) \| \| x(s-\sigma) \| \Delta s
\leq \| P \| + \sum_{i=0}^{\infty} \int_{t_1 + (2i-1)\tau}^{t_1 + 2i\tau} \| Q(s) \| \| x(s-\sigma) \| \Delta s
\leq \| P \| + p_2 \sum_{i=0}^{\infty} \int_{t_1 + (2i-1)\tau}^{t_1 + 2i\tau} \| Q(s) \| \Delta s
\leq \| P \| + p_2 \frac{\| P \| - p_1}{p_2}
= 2\| P \| - p_1
\leq p_2.
\]
Further,
\[
\|(Fx)(t)\| = \|P + \sum_{i=0}^{\infty} \int_{t+i(2i-1)\tau}^{t+2i\tau} Q(s)x(s-\sigma)\Delta s\|
\]
\[
\geq \|P\| - \|\sum_{i=0}^{\infty} \int_{t+i(2i-1)\tau}^{t+2i\tau} Q(s)x(s-\sigma)\Delta s\|
\]
\[
\geq \|P\| - \sum_{i=0}^{\infty} \int_{t+i(2i-1)\tau}^{t+2i\tau} Q(s)x(s-\sigma)\Delta s\|
\]
\[
\geq \|P\| - \sum_{i=0}^{\infty} \int_{t+i(2i-1)\tau}^{t+2i\tau} \|Q(s)\|\|x(s-\sigma)\|\Delta s
\]
\[
\geq \|P\| - p_2 \sum_{i=0}^{\infty} \int_{t+i(2i-1)\tau}^{t+2i\tau} \|Q(s)\|\Delta s
\]
\[
= \frac{\|P\| - p_1}{p_2}
\]
\[
= p_1.
\]
Thus we proved that \(FA \subset A\).

Since \(A\) is a bounded, closed and convex subset of \(X\), we prove that \(T\) is a contraction mapping on \(A\). Now, for all \(x_1, x_2 \in A\), and \(t \geq t_1\), we have
\[
\|(Fx_1)(t) - (Fx_2)(t)\| = \|\sum_{i=0}^{\infty} \int_{t+i(2i-1)\tau}^{t+2i\tau} Q(s)[x_1(s-\sigma) - x_2(s-\sigma)]\Delta s\|
\]
\[
\leq \sum_{i=0}^{\infty} \int_{t+i(2i-1)\tau}^{t+2i\tau} \|Q(s)\|[x_1(s-\sigma) - x_2(s-\sigma)]\|\Delta s
\]
\[
\leq \sum_{i=0}^{\infty} \int_{t+i(2i-1)\tau}^{t+2i\tau} \|Q(s)\||[x_1(s-\sigma) - x_2(s-\sigma)]\|\Delta s
\]
\[
\leq \|x_1 - x_2\| \sum_{i=0}^{\infty} \int_{t+i(2i-1)\tau}^{t+2i\tau} \|Q(s)\|\Delta s
\]
\[
= q_3\|x_1 - x_2\|.
\]
This immediately implies that
\[
\|(Fx_1)(t) - (Fx_2)(t)\| \leq q_3\|x_1 - x_2\|,
\]
where
\[
q_3 \leq \sum_{i=0}^{\infty} \int_{t+i(2i-1)\tau}^{t+2i\tau} \|Q(s)\|\Delta s \leq \frac{\|P\| - p_1}{p_2} < 1.
\]
which proves that \(F\) is a contraction mapping. Consequently, \(F\) has the fixed point \(x\) with \(Fx = x\) with \(|x| > 0\), for all \(t \geq t_1\), which is a nonoscillatory solution of Eq. (1) which completes the proof of Case 3.

**Case 4.** \(-1 < p < 0\).

Choose a \(t_1 \in \mathbb{T}\) sufficiently large such that \(t_1 \geq t_0 + \max\{\tau, \sigma\}\), and
\[
\int_{t_1}^{\infty} \|Q(s)\|\Delta s \leq \frac{1 + p(1 + L_2) - L_1}{L_2}.
\]
hold, where \(0 < L_1 < 1, L_1, L_2\) are positive constants such that
\[
2(1 + p) < L_1 + L_2 < 2, \quad \frac{L_1 - 1}{1 + L_2} < p \leq \frac{L_1 + L_2}{2} - 1.
\]

Let \(X(t)\) be the set of all rd–continuous and bounded vector functions on \(t_0 \leq t \in \mathbb{T}\). Set \(A = \{x \in X : L_1 \leq \|x(t)\| \leq L_2, t_0 \leq t \leq t\}\). Define a mapping \(F : A \to X\) as follows
\[
(Fx)(t) = \begin{cases} 
(1 + p)e - px(t - \tau) + \int_t^\infty Q(s)x(s - \sigma)\Delta s, & t \geq t_1; \\
(Fx)(t_1), & t_0 \leq t \leq t_1.
\end{cases}
\]
Clearly \(Fx\) is rd–continuous. For every \(x \in A\) and \(t \geq t_1\), we have
\[
\|(Fx)(t)\| = \|(1 + p)e - px(t - \tau) + \int_t^\infty Q(s)x(s - \sigma)\Delta s\|
\leq \|(1 + p)e\| + \|px(t - \tau)\| + \|\int_t^\infty Q(s)x(s - \sigma)\Delta s\|
\leq (1 + p) - p\|x(t - \tau)\| + \int_t^\infty \|Q(s)x(s - \sigma)\|\Delta s
\leq 1 + p - pL_2 + L_2 \int_t^\infty \|Q(s)\|\Delta s
\leq 1 + p - pL_2 + L_2 \frac{1 + p(1 + L_2) - L_1}{L_2}
= 2(1 + p) - L_1
\leq L_2.
\]

Further
\[
\|(Fx)(t)\| = \|(1 + p)e - [px(t - \tau) - \int_t^\infty Q(s)x(s - \sigma)\Delta s]\|
\geq \|(1 + p)e\| - \|px(t - \tau) - \int_t^\infty Q(s)x(s - \sigma)\Delta s\|
\geq (1 + p)\|e\| - \| - px(t - \tau)\| - \|\int_t^\infty Q(s)x(s - \sigma)\|\Delta s
\geq (1 + p) + p\|x(t - \tau)\| - \int_t^\infty \|Q(s)x(s - \sigma)\|\Delta s
\geq 1 + p + pL_2 - \int_t^\infty \|Q(s)\|\|x(s - \sigma)\|\Delta s
\geq 1 + p + pL_2 - L_2 \int_t^\infty \|Q(s)\|\Delta s
\geq 1 + p + pL_2 - L_2 \frac{1 + p(1 + L_2) - L_1}{L_2}
= L_1.
\]

Thus we proved that \(FA \subset A\).

Since \(A\) is a bounded, closed and convex subset of \(X\), we prove that \(F\) is a contraction mapping on \(A\).
Now, for all $x_1, x_2 \in A$, and $t \geq t_1$, we have

$$
\| (Fx_1)(t) - (Fx_2)(t) \| \\
= \| -p[x_1(t-\tau) - x_2(t-\tau)] + \int_t^\infty Q(s)[x_1(s-\sigma) - x_2(s-\sigma)]\Delta s \| \\
\leq \| -p[x_1(t-\tau) - x_2(t-\tau)] \| + \| \int_t^\infty Q(s)[x_1(s-\sigma) - x_2(s-\sigma)]\Delta s \| \\
\leq -p\| x_1 - x_2 \| + \| x_1 - x_2 \| \int_t^\infty \| Q(s) \| \Delta s \\
\leq q_4\| x_1 - x_2 \| \quad (q_4 = p + \int_t^\infty \| Q(s) \| \Delta s).
$$

This immediately implies that

$$
\| (Fx_1)(t) - (Fx_2)(t) \| \leq q_4\| x_1 - x_2 \|.
$$

where

$$
q_4 = -p + \int_t^\infty \| Q(s) \| \Delta s \leq -p + \frac{1 + p(1 + L_2) - L_1}{L_2} = \frac{1 + p - L_1}{L_2} < 1.
$$

which proves that $F$ is a contraction mapping. Consequently, $F$ has the fixed point $x$ with $Fx = x$, $\|x\| > 0$, for all $t \geq t_1$, which is a nonoscillatory solution of Eq.(1) which completes the proof of Case 4.

**Case 5.** $-\infty < p < -1$.

Choose a $t_1 \in \mathbb{T}$ sufficiently large such that $t_1 + \tau \geq t_0 + \sigma$, and

$$
\int_t^\infty \| Q(s) \| \Delta s \leq \frac{pK_1 - 1 - p - K_2}{K_2}.
$$

where $0 < K_1 < K_2 < 1$, $K_1 + K_2 > 1$, and $\frac{2}{K_1 + K_2 - 2} < p < \frac{1 + K_2}{K_1 - 1}$.

Let $X(t)$ be the set of all rd-continuous and bounded vector functions on $t_0 \leq t \in \mathbb{T}$. Set $A = \{x \in X : K_1 \leq \|x(t)\| \leq K_2, t_0 \leq t \in \mathbb{T}\}$. Define a mapping $F : A \to X$ as follows

$$
(Fx)(t) = \begin{cases} 
(1 + \frac{1}{p})e - \frac{1}{p}x(t+\tau) + \frac{1}{p}\int_t^\infty Q(s)x(s-\sigma)\Delta s, & t \geq t_1; \\
(Fx)(t_1), & t_0 \leq t \leq t_1.
\end{cases}
$$
Clearly $Fx$ is rd–continuous. For every $x \in A$ and $t \geq t_1 \in \mathbb{T}$, we have
\[
\|(Fx)(t)\| = \|(1 + \frac{1}{p})e - \frac{1}{p}x(t + \tau) + \frac{1}{p} \int_t^\infty Q(s)x(s - \sigma)\|s\|
\]
\[
\leq \|(1 + \frac{1}{p})e\| + \|\frac{1}{p}x(t + \tau)\| + \frac{1}{p} \int_t^\infty Q(s)x(s - \sigma)\|s\|
\]
\[
\leq (1 + \frac{1}{p})\|e\| - \frac{1}{p}\|x(t + \tau)\| - \frac{1}{p} \int_t^\infty Q(s)x(s - \sigma)\|s\|
\]
\[
\leq 1 + \frac{1}{p}\|x(t + \tau)\| + \frac{1}{p} \int_t^\infty \|Q(s)x(s - \sigma)\|s\|
\]
\[
\leq 1 + \frac{1}{p} - \frac{K_2}{p} - \frac{K_2}{p} \int_t^\infty \|Q(s)\|s\|
\]
\[
= 2(1 + \frac{1}{p}) - K_1
\]
\[
\leq K_2.
\]

Further,
\[
\|(Fx)(t)\| = \|(1 + \frac{1}{p})e - \frac{1}{p}x(t + \tau) - \frac{1}{p} \int_t^\infty Q(s)x(s - \sigma)\|s\|
\]
\[
\geq \|(1 + \frac{1}{p})\| - \|\frac{1}{p}x(t + \tau)\| - \frac{1}{p} \int_t^\infty Q(s)x(s - \sigma)\|s\|
\]
\[
\geq 1 + \frac{1}{p} + \frac{1}{p}\|x(t + \tau)\| + \frac{1}{p} \int_t^\infty \|Q(s)x(s - \sigma)\|s\|
\]
\[
\geq 1 + \frac{1}{p} + \frac{K_2}{p} + \frac{1}{p} \int_t^\infty \|Q(s)\|s\|
\]
\[
\geq 1 + \frac{1}{p} + \frac{K_2}{p} + \frac{K_2}{p} \int_t^\infty \|Q(s)\|s\|
\]
\[
\geq 1 + \frac{1}{p} + \frac{K_2}{p} + \frac{K_2 pK_1 - 1 - p - K_2}{K_2}
\]
\[
= K_1.
\]

Thus we proved that $FA \subset A$.

Since $A$ is a bounded, closed and convex subset of $X$, we prove that $F$ is a contraction mapping on $A$. Now, for all $x_1, x_2 \in A$, and $t \geq t_1$, we have
\[
\|(Fx_1)(t) - (Fx_2)(t)\|
\]
\[
= \|\frac{1}{p}[x_1(t + \tau) - x_2(t + \tau)] - \frac{1}{p} \int_t^\infty Q(s)[x_1(s - \sigma) - x_2(s - \sigma)]\|s\|
\]
\[
\leq \|\frac{1}{p}[x_1(t + \tau) - x_2(t + \tau)]\| + \|\frac{1}{p} \int_t^\infty Q(s)[x_1(s - \sigma) - x_2(s - \sigma)]\|s\|
\]
\[
\leq - \frac{1}{p}\|x_1(t + \tau) - x_2(t + \tau)\| + \frac{1}{p} \int_t^\infty \|Q(s)\|s\|
\]
\[
\leq - \frac{1}{p}\|x_1 - x_2\| - \frac{1}{p}\|x_1 - x_2\| \int_t^\infty \|Q(s)\|s\|
\]
\[
\leq q_5\|x_1 - x_2\| \ (q_5 = -\frac{1}{p} + \int_t^\infty \|Q(s)\|s\|).
\]
This immediately implies that
\[ q_5 = \frac{1}{p} + \frac{1}{p} \int_t^\infty \|Q(s)\| \Delta s \leq \frac{1 + p - pK_1}{pK_2} < 1. \]

Therefore,
\[ \|(Fx_1)(t) - (Fx_2)(t)\| \leq q_5 \|x_1 - x_2\|, \]
which proves that \( F \) is a contraction mapping. Consequently, \( F \) has the fixed point \( x \) with \( \|x\| > 0 \), for all \( t \geq t_1 \), which is a nonoscillatory solution of Eq. (1), which completes the proof of Case 5.

Now we consider \( B \) is a nonsingular constant matrix. Assume that \( \|B\| = b \)

**Theorem 2.2.** Suppose that
\[ \int_{t_1}^\infty \|Q(s)\| \Delta s < \infty, \]
Then Eq.(2) has a nonoscillatory solution.

**Proof.** **Case 1.** \( b \in [0,1) \)

Choose a \( t_1 \in \mathbb{T} \) sufficiently large such that \( t_1 \geq t_0 + \bar{\sigma}, \ \bar{\sigma} = \max \tau, \sigma, \) and
\[ \int_{t_1}^\infty \|Q(s)\| \Delta s \leq \frac{1 - b(1 + M_2) - M_1}{M_2} \]
hold. where \( 0 < M_1 < 1, 0 < M_2, \) \( 1 - b < M_1 + M_2 < 2, 1 - \frac{M_1 + M_2}{2} \leq b < \frac{1 - M_1}{1 + M_2}. \)

Let \( X(t) \) be the set of all \( rd \)-continuous and bounded vector functions on \( t_0 \leq t \leq t_1 \). Set \( A = \{ x \in X : M_1 \leq \|x(t)\| \leq M_2, t_0 \leq t \leq t_1 \} \). Define a mapping \( F : A \rightarrow X \) as follows
\[
(Fx)(t) = \begin{cases} 
q - Bx(t - \tau) + \int_t^\infty Q(s)x(s - \sigma)\Delta s, & t \geq t_1; \\
(Fx)(t_1), & t_0 \leq t \leq t_1.
\end{cases}
\]
where \( q \) is a vector such that \( \|q\| = 1 - b \).

Clearly \( Fx \) is \( C_{rd} \) continuous. For every \( x \in A \) and \( t \geq t_1 \), we get
\[
\|(Fx)(t)\| = \|q - Bx(t - \tau) + \int_t^\infty Q(s)x(s - \sigma)\Delta s\|
\leq \|q\| + \|Bx(t - \tau)\| + \|\int_t^\infty Q(s)x(s - \sigma)\Delta s\|
\leq (1 - b) + b\|x(t - \tau)\| + \int_t^\infty \|Q(s)x(s - \sigma)\| \Delta s
\leq 1 - b + bM_2 + M_2 \int_t^\infty \|Q(s)\| \Delta s
\leq 1 - b + bM_2 + M_2 \frac{1 - b(1 + M_2) - M_1}{M_2}
= 2(1 - b) - M_1
\leq M_2.
\]
Further,

\[
\|(Fx)(t)\| = \|q - [Bx(t - \tau) - \int_{t}^{\infty} Q(s)x(s - \sigma)\Delta s]\|
\geq \|q\| - \|Bx(t - \tau) - \int_{t}^{\infty} Q(s)x(s - \sigma)\Delta s\|
\geq 1 - b - b\|x(t - \tau)\| - \int_{t}^{\infty} \|Q(s)x(s - \sigma)\|\Delta s
\geq 1 - b - bM_2 - M_2 \int_{t}^{\infty} \|Q(s)\|\Delta s
\geq 1 - b - bM_2 - \frac{M_1}{M_2}
= M_1.
\]

Thus we proved that \(FA \subset A\).

Since \(A\) is a bounded, closed and convex subset of \(X\) we have to prove that \(F\) is a contraction mapping on \(A\) in order to apply the contraction principle. Now, for all \(x_1, x_2 \in A, t \geq t_1\), we have

\[
\|(Fx_1)(t) - (Fx_2)(t)\|
= \| - B[x_1(t - \tau) - x_2(t - \tau)] + \int_{t}^{\infty} Q(s)[x_1(s - \sigma) - x_2(s - \sigma)]\Delta s\|
\leq \| - B[x_1(t - \tau) - x_2(t - \tau)]\| + \| \int_{t}^{\infty} Q(s)[x_1(s - \sigma) - x_2(s - \sigma)]\Delta s\|
\leq b\|x_1 - x_2\| + \|x_1 - x_2\| \int_{t}^{\infty} \|Q(s)\|\Delta s
\leq r_1\|x_1 - x_2\| \quad (r_1 = b + \int_{t}^{\infty} \|Q(s)\|\Delta s).
\]

Clearly,

\[
r_1 = b + \int_{t}^{\infty} \|Q(s)\|\Delta s \leq b + \frac{1 - b(1 + M_2) - M_1}{M_2} = \frac{1 - b - M_1}{M_2} < 1.
\]

This immediately implies that

\[
\|(Fx_1)(t) - (Fx_2)(t)\| \leq r_1\|x_1 - x_2\|,
\]

which proves that \(F\) is a contraction mapping. Consequently, \(F\) has the fixed point \(x\) with \(Fx = x, \|x\| > 0\), for all \(t \geq t_1\), which is a nonoscillatory solution of Eq. (2) which completes the proof of Case 1.

**Case 2.** \(1 < b < \infty\).

Choose a \(t_1 \in \mathbb{T}\) sufficiently large such that \(t_1 \geq t_0 + \overline{\sigma}\), and

\[
\int_{t_1 + \tau}^{\infty} \|Q(s)\|\Delta s \leq \frac{b - 1 - pN_1 - N_2}{N_2}
\]

hold, where \(0 < N_1 < 1, N_2 > N_1\) and \(N_1 + N_2 < 2\), \(\frac{1 + N_2}{1 - N_1} < b \leq \frac{2}{2 - N_1 - N_2}\).
Let $X(t)$ be the set as in Case 1. Set $A = \{ x \in X : N_1 \leq \| x(t) \| \leq N_2, \; t_0 \leq t \in T \}$. Define a mapping $F : A \rightarrow X$ as follows

$$(Fx)(t) = \begin{cases} r - B^{-1}x(t - \tau) + B^{-1} \int_{t}^{\infty} Q(s)x(s) \Delta s, & t \geq t_1; \\ (Fx)(t_1), & t_0 \leq t \leq t_1. \end{cases}$$

where $r$ is a vector such that $\|r\| = 1 - \frac{1}{b}$.

Clearly $Fx$ is rd continuous. For every $x \in A$ and $t \geq t_1$, we get

$$\| (Fx)(t) \| = \| r - B^{-1}x(t - \tau) + B^{-1} \int_{t}^{\infty} Q(s)x(s) \Delta s \| \leq \| r \| + \| B^{-1}x(t - \tau) \| + \| B^{-1} \int_{t}^{\infty} Q(s)x(s) \Delta s \| \leq (1 - \frac{1}{b}) + \frac{1}{b} \| x(t - \tau) \| + \frac{1}{b} \| \int_{t}^{\infty} Q(s)x(s) \| \Delta s \leq 1 - \frac{1}{b} + \frac{N_2}{b} + \frac{N_2}{b} \int_{t}^{\infty} \| Q(s) \| \Delta s \leq 1 - \frac{1}{b} + \frac{N_2}{b} + \frac{N_2}{b} (1 - bN_1 - N_2) = 2(1 - \frac{1}{b}) - N_1 \leq N_2.$$ 

Further

$$\| (Fx)(t) \| = \| r - [B^{-1}x(t - \tau) - B^{-1} \int_{t}^{\infty} Q(s)x(s) \Delta s] \| \geq \| (1 - \frac{1}{b})e \| - \| B^{-1}x(t - \tau) - B^{-1} \int_{t}^{\infty} Q(s)x(s) \Delta s \| \geq 1 - \frac{1}{b} - \frac{1}{b} \| x(t - \tau) \| - \frac{1}{b} \int_{t}^{\infty} \| Q(s) \| \| x(s - \sigma) \| \Delta s \geq 1 - \frac{1}{b} - \frac{N_2}{b} - \frac{1}{b} \int_{t}^{\infty} \| Q(s) \| \| x(s - \sigma) \| \Delta s \geq 1 - \frac{1}{b} - \frac{N_2}{b} - \frac{N_2}{b} \int_{t}^{\infty} \| Q(s) \| \Delta s \geq 1 - \frac{1}{b} - \frac{N_2}{b} - \frac{N_2}{b} \int_{t}^{\infty} \| Q(s) \| \Delta s \geq 1 - \frac{1}{b} - \frac{N_2}{b} - \frac{N_2}{b} (1 - bN_1 - N_2) = N_2.$$ 

Thus we proved that $FA \subset A$.

Since $A$ is a bounded, closed and convex subset of $X$, we have to prove that $F$ is a contraction mapping on $A$ in order to apply the contraction principle. Now, for
all $x_1, x_2 \in A, t \geq t_1$, we have

$$
\|(Fx_1)(t) - (Fx_2)(t)\|
\leq \| - B^{-1}[x_1(t - \tau) - x_2(t - \tau)] + B^{-1}\int_{t}^{\infty} Q(s)[x_1(s - \sigma) - x_2(s - \sigma)]\Delta s \|
\leq \| - B^{-1}[x_1(t - \tau) - x_2(t - \tau)] + B^{-1}\int_{t}^{\infty} Q(s)[x_1(s - \sigma) - x_2(s - \sigma)]\Delta s \|
\leq \frac{1}{b}\|x_1(t - \tau) - x_2(t - \tau)\| + \frac{1}{b}\int_{t}^{\infty} \|Q(s)\|\|x_1(s - \sigma) - x_2(s - \sigma)\|\Delta s
\leq \frac{1}{b}\|x_1 - x_2\| + \frac{1}{b}\|x_1 - x_2\| \int_{t}^{\infty} \|Q(s)\|\Delta s
\leq r_2\|x_1 - x_2\| (r_2 = \frac{1}{b} + \frac{1}{b}\int_{t}^{\infty} \|Q(s)\|\Delta s).

Clearly,

$$
r_2 = \frac{1}{b} + \frac{1}{b}\int_{t}^{\infty} \|Q(s)\|\Delta s \leq \frac{1}{b} + \frac{b - 1 - bN_1 - N_2}{N_2} = \frac{b - 1 - bN_1}{pN_2} < 1.
$$

This immediately implies that

$$
\|(Fx_1)(t) - (Fx_2)(t)\| \leq r_2\|x_1 - x_2\|,
$$

which proves that $F$ is a contraction mapping. Consequently, $F$ has the fixed point $x$ with $Fx = x, \|x\| > 0$, for all $t > t_1$, which is a nonoscillatory solution of Eq.(2) which completes the proof of Case 2.

REFERENCES