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European Option Pricing Formula Under Stochastic Interest Rate

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Abstract: This paper reviews the option pricing model and its application, on the basis of former studies, we assume that the interest rate satisfy a given Vasicek stochastic differential equation, using option pricing by martingale method to study the stochastic interest rate model of European option pricing and obtain its pricing formula. Finally, we compare the differences between the standard European option pricing formula and European option pricing formula under stochastic interest rate.

Key words: Option pricing; Stochastic interest rates; Vasicek model; Brownian motions

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1. INTRODUCTION

Black, Scholes (1973) and Merton [2](1973) showed in their seminal papers that a derivative security can be priced by creating a replicating portfolio, i. e., a portfolio of primitive securities which matches the payoff of the derivative at maturity. Since both the replication portfolio and the derivative offer the same payoff at maturity, they must have the same price at any exercise time. Deviations from this equality lead to arbitrage possibilities. Hence, the pricing by duplication procedure inhibits arbitrage by construction. Since then the field of financial engineering has grown phenomenally. The BlackCScholesCMerton risk neutrality formulation of the option pricing theory is attractive because the pricing formula of a derivative deduced from their model is a function of several directly observable parameters (except one, which is the volatility parameter). The derivative can be priced as if the market price of the underlying assets risk is zero.

Detemple [8] (2005) reviewed the valuation of American options. Several semi-analytical approximations for American option prices have been proposed in the literature (Barone Adesi & Whaley [1], 1987; Broadie & Detemple [5], 1996; Bunch & Johnson [6], 2000). Although these approaches are fast and accurate, they can not easily be extended beyond the Black-Scholes model. It has been firmly established that the Black-Scholes model is not consistent with quoted option prices. The literature advocates the introduction of stochastic volatility or jump store produce the implied volatility smile observed in the market. The introduction of an additional stochastic volatility factor enormously complicates the pricing of American options. Presently, this can only be done by means of numerical schemes, which involve solving integral equations(Kim [16], 1990; Huang, Subrahmanyam & Yu [13], 1996; Sullivan [20], 2000; Detemple & Tian [9], 2002), performing Monte Carlo simulations(Broadie & Glasserman [4], 1997; Longstaff & Schwartz [17], 2001; Rogers [19], 2002; Haugh & Kogan [12], 2004), or discrete the partial differential equation(Brennan & Schwartz [3], 1977; Clarke & Parrott [7], 1999; Ikonen & Toivanen [15], 2007). The early exercise premium of the American put option depends on the cost of carry determined by interest rates. Consequently, the volatility of interest rates does affect the decision to exercise this option at any point in time. This fact is recognized in the literature dealing with models with stochastic interest rates (Ho, Stapleton & Subrahmanyam [11], 1997; Menkveld & Vorst [18], 2001; Detemple & Tian [9], 2002). This literature, however, considers only two-factor extensions of the Black-Scholes model assuming that the volatility of the underlying asset is constant.

In this paper, we assume that the interest rate subject to a given Vasicek stochastic differential equations, by using martingale method to study the stochastic interest rate model of European option pricing and obtain the pricing formula.

The paper is organized as follows. In Section 2 we describe the assumptions of the option model, using martingale method, by solving a second order parabolic partial differential equation, we obtain the European option pricing formula. In Section 3 we compare the differences between the standard European option pricing formulas and European option pricing formula under stochastic interest rate.

2. EUROPEAN OPTION PRICING FORMULA

The standard BS model makes the following assumptions: the market is frictionless (i. e., no transaction costs or taxes and no penalties for short selling); the market operates continuously, the risk-free interest rate r is a known constant; the asset price X_t follows Geometrical Brownian Motion (GBM) with constant volatility $\sigma > 0$ and pays no dividends; options and derivatives are European (i. e., no early exercise) and expire at time T with a payoff that depends only on X_T; the market is arbitrage free.

Under the assumption of GBM, the asset price X_t satisfies a stochastic differential equation (sde) of the form

$$dX_t = X_t(\mu dt + \sigma dB_t), X_0 = x_0$$

where μ is the growth rate of the asset. The term B_t is a standard Brownian motion under a measure P (called the real-world measure).

Theorem 1 (Itôs Lemma) Let X_t satisfy the sde

$$dX_t = \alpha(xt)dt + \beta(x,t)dB_t$$

where $x = X_t$ and let $V(x,t)$ be any $C_{2,1}$ function. Then $V(X_t,t)$ satisfies the sde

$$dV = \left[V_t + \alpha V_x + \frac{1}{2}\beta^2 V_{xx}\right]dt + \beta V_x dB_t$$

where subscripts on V denote partial derivatives. It is sometimes more instructive to write this last sde in the equivalent form

$$dV = \left[V_t + \frac{1}{2}\beta^2 V_{xx}\right]dt + V_x dX_t$$

To solve sde, the main method is use Itô**s** Lemma. For example, the sde (1) for GBM can be solved by taking $Y_t(X_t) = log(\frac{X_t}{x_0})$. The sde for Y_t then becomes $dY_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dB_t$, and this is readily integrated to give the representation

$$X_t \stackrel{d}{=} x_0 \exp\{(\mu - \frac{1}{2}\sigma^2)t\} + \sigma\sqrt{t}Z, Z \sim N(0,1)$$

Now, lets deduce the euro-options pricing formula under stochastic interest rate. Assume the asset price X_t satisfies GBM

$$\frac{dX_t}{X_t} = r_t dt + \sigma_1(r_t,t)dB_t^1 \tag{1}$$

The interest rate is given by Vasicek Model [21]

$$dr_t = a(\theta - r_t)dt + \sigma_2(r_t,t)dB_t^2 \tag{2}$$

where $\{B_t^1 : t \geq 0\}, \{B_t^2 : t \geq 0\}$ are standard Brown motions,

$$cov(dB_t^1, dB_t^2) = \rho dt, (|\rho| < 1) \tag{3}$$

Let $V_t = V(X_t, r_t, t)$ denote the price of the call European option, $V_t = (X_t - K)^+$, K is the strike price.

Now, we need to find the option price. Using $\Delta-$ hedging technical we derive function $V(X_t, r_t, t)$ satisfy the appropriate sde, and obtain the portfolio

$$\prod_t = V_t - \Delta_{1t}X_t - \Delta_{2t}P_t$$

Choose Δ_{1t} share of stock and Δ_{2t} share of zero-coupon, the portfolio $\prod_t$ is risk-free in the period $[t, t+dt]$. Its also mean if choose appropriate Δ_{1t} and Δ_{2t}, then we can get

$$d\prod_t = dV_t - \Delta_{1t}dX_t - \Delta_{2t}dP_t \tag{4}$$

which is risk-free, and then

$$d\prod_t = r_t \prod_t dt = r_t\left[V_t - \Delta_{1t}X_t - \Delta_{2t}P_t\right]dt \tag{5}$$

Here $P_t = P(r_t, t; T)$ is the price of zero-coupon, and satisfies a stochastic differential equation (sde) of the form

$\frac{dP_t}{P_t} = r_t dt - X_V(t)\sigma dB_t$ (Vasicek Model)

or $\frac{dP_t}{P_t} = r_t dt - X_C(t)\sigma\sqrt{r_t}dB_t$ (C-I-R Model).

By Itôs Lemma, (4) can be written as

$$d\prod_t = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 X^2 \frac{\partial^2 V}{\partial X^2} + \sigma_1\sigma_2\rho X\frac{\partial^2 V}{\partial X\partial r} + \frac{1}{2}\sigma_2^2\frac{\partial^2 V}{\partial r^2}\right] dt + \left(\frac{\partial V}{\partial X} - \Delta_{1t}\right) \mathrm{dX_t} + \left(\frac{\partial V}{\partial r} - \Delta_{2t}\frac{\partial P}{\partial r}\right) dr_t - \Delta_{2t}\left(\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_2^2\frac{\partial^2 P}{\partial r^2}\right) dt \tag{6}$$

The third term of the right equation can be substitute as

$$-\Delta_{2t}\left(rP - a(\theta - r\frac{\partial P}{\partial r})\right) dt \tag{7}$$

In order to eliminate the risk, let

$$\frac{\partial V}{\partial X} = \Delta_{1t}, \Delta_{2t} = \frac{\partial V}{\partial t} / \frac{\partial P}{\partial r}$$

considering (7) and (2), we obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 X^2\frac{\partial^2 V}{\partial X^2} + \sigma_1\sigma_2\rho X\frac{\partial^2 V}{\partial X\partial r} + \frac{1}{2}\sigma_2^2\frac{\partial^2 V}{\partial r^2} + rX\frac{\partial V}{\partial X} + a(\theta - r)\frac{\partial V}{\partial r} - rV = 0, \big(r \in \mathbb{R}, X \in \mathbb{R}^+, t \in [0, T]\big) \tag{8}$$

as

$$t = T, V(X, t, T) = (X - K)^+, (r \in \mathbb{R}, X \in \mathbb{R}^+) \tag{9}$$

Since the martingale method option pricing theory [10], there exists a martingale measure Q, such that

$$V_t = E^Q\big(e^{-\int_t^T r_\tau d\tau}(X_T - K)^+ | r(t) = r_t, X(t) = X_t\big) \tag{10}$$

To transform the account unit valuation, we use zero-coupon $P_t = P(r_t, t; T)$ as a new account unit, and a corresponding price system introduced

$$\widehat{X_t} = \frac{X_t}{P_t}, \widehat{V_t} = \frac{V_t}{P_t} \tag{11}$$

Then the equivalent martingale measure exists, such that (10) can be rewritten as

$$\begin{aligned} V_t &= E^{Q^U}\left(\widehat{V}_t | r(t) = r_t, X(t) = X_t\right) \\ &= E^{Q^U}\left(\left(\frac{X_T}{P_T} - K\right)^+ \middle| P(t) = P_t, X(t) = X_t\right) \\ &= E^{Q^U}\left(\left(\widehat{X}_T - K\right)^+ | P(t) = P_t, X(t) = X_t\right) \end{aligned} \tag{12}$$

In the last equation, we use the fact $P_T = 1$.

Equation (12) indicates at the time T, function $\widehat{V}_t$ can be transformed only depends on X_t. In the Vasicek model, zero-coupon price processes satisfies stochastic differential equation, $\dfrac{dP_t}{P_t} = r_t dt - M(t)\sigma dB_t$, so,

$$d\ln P_t = \left[r_t - \frac{1}{2}\sigma_2^2 X_V^2(t)\right] dt - \sigma_2 X_V(t) dB_t^2$$

According to (1),

$$d\ln X_t = \left[r_t - \frac{1}{2}\sigma_1^2\right] dt + \sigma_1 dB_t^2$$

As for $\widehat{X_t} = \frac{X_t}{P_t}$, such that

$$d\ln\widehat{X_t} = d\ln X_t - d\ln P_t = \frac{1}{2}\left[\sigma_2^2 X_V^2(t) - \sigma_1^2\right] dt + \sigma_1 dB_t^1 + \sigma_2 X(t) dB_t^2$$

This formula indicates that for the Vasicek model, the stochastic differential equations of $\widehat{X_t}$ is no longer significant contain with X_t and r_t , so we can put (12) rewritten as

$$\widehat{V}_t = E^{Q^U}\left(\left(\widehat{X}_T - K\right)^+ \middle| \widehat{X}(t) = X_t\right)$$

This shows that for the new price system $\{\widehat{X_t}, \widehat{V_t}\}, \widehat{V}_t = \widehat{V}(\widehat{X_t}, t)$ holds.

To solve problem (8) and (9), we draw a new transformation of independent variables,

$$y = \frac{X}{P(r,t;T)} \tag{13}$$

and a new unknown function denotes as

$$\widehat{V}(y,t) = \frac{V(X,r,t)}{P(r,t;T)} \tag{14}$$

According to primarily computations,

$$\begin{aligned}
\frac{\partial V}{\partial t} &= \widehat{V}\frac{\partial P}{\partial t} + P\frac{\partial \widehat{V}}{\partial t} - y\frac{\partial \widehat{V}}{\partial y}\frac{\partial P}{\partial t} \\
\frac{\partial V}{\partial r} &= \widehat{V}\frac{\partial P}{\partial r} - y\frac{\partial \widehat{V}}{\partial y}\frac{\partial P}{\partial r} \\
\frac{\partial V}{\partial X} &= \frac{\partial \widehat{V}}{\partial y} \\
\frac{\partial^2 V}{\partial r^2} &= \widehat{V}\frac{\partial^2 P}{\partial r^2} - y\frac{\partial \widehat{V}}{\partial y}\frac{\partial^2 P}{\partial r^2} - y^2\frac{\partial^2 \widehat{V}}{\partial y^2}\frac{1}{P}\left(\frac{\partial P}{\partial r}\right)^2 \\
\frac{\partial^2 V}{\partial r \partial X} &= -y\frac{\partial^2 \widehat{V}}{\partial y^2}\frac{1}{P}\frac{\partial P}{\partial r} \\
\frac{\partial^2 V}{\partial X^2} &= \frac{1}{P}\frac{\partial^2 \widehat{V}}{\partial y^2}
\end{aligned}$$

Substitute them into (8), and divided by $P(r,t;T)$, such that

$$\begin{aligned}&\frac{\partial \widehat{V}}{\partial t}+\frac{1}{2}\left[\sigma_1^2\frac{X^2}{P^2}-2\sigma_1\sigma_2\rho\frac{X}{P}\frac{\partial P}{\partial r}+\frac{1}{2}\sigma_2^2y^2\left(\frac{1}{P}\frac{\partial P}{\partial r}\right)^2\right]\frac{\partial^2\widehat{V}}{\partial y^2}\\&+\frac{1}{P}\left[\frac{\partial P}{\partial t}+\frac{\sigma_2^2}{2}\frac{\partial^2 P}{\partial r^2}+a(\theta-r)\frac{\partial P}{\partial r}-r\frac{X}{y}\right]y\frac{\partial\widehat{V}}{\partial y}\\&+\frac{1}{P}\left[\frac{\partial P}{\partial t}+\frac{\sigma_2^2}{2}\frac{\partial^2 P}{\partial r^2}+a(\theta-r)\frac{\partial P}{\partial r}-rP\right]\widehat{V}=0\end{aligned}$$

Considering transformation (13) and the function $P(r,t;T)$ satisfies the following second order parabolic pdes Cauchy problem

$$\begin{cases}\frac{\partial P}{\partial t}+\frac{\sigma^2}{2}\frac{\partial^2 P}{\partial r^2}+a(\theta-r)\frac{\partial P}{\partial r}-rP=0,(r\in\mathbb{R},t\in[0,T])\\ P(r,T)=1.\end{cases}\tag{15}$$

Then, we immediately find that function $\widehat{V}(y,t)$ satisfies the equation

$$\frac{\partial\widehat{V}}{\partial r}+\frac{1}{2}\widehat{\sigma}^2(t)y^2\frac{\partial^2\widehat{V}}{\partial y^2}=0$$

and the definitely solution condition is

$$\widehat{V}(y,T)=\frac{V(X,r,T)}{P(r,T;T)}=(y-K)^+$$

where K is the options strike price,

$$\widehat{\sigma}(t)=\sqrt{\sigma_1^2+2\rho\sigma_1\sigma_2X(t)+\sigma_2^2X^2(t)},X(t)=-\frac{1}{P_t}\frac{\partial P}{\partial r}$$

The solution of problem (15) can be expressed by the general Black-Scholes formula

$$\widehat{V}(y,t)=yN(d_1)-KN(d_2)\tag{16}$$

$$d_1=\frac{\ln\frac{y}{K}+\frac{1}{2}\int_t^T\widehat{\sigma}^2(\tau)d\tau}{\sqrt{\int_t^T\widehat{\sigma}^2(\tau)d\tau}}\tag{17}$$

$$d_2=d_1-\sqrt{\int_t^T\widehat{\sigma}^2(\tau)d\tau}\tag{18}$$

Reverse to the original variables X,r,t and unknown function V by the transformation (13) and (14), then (16) and (17) become

$$V(X,r,t)=P(r,t;T)\widehat{V}\left(\frac{X}{P(r,t;T)},t\right)=\mathrm{X}N(d_1^*)-KP(r,t;T)N(d_2^*)\tag{19}$$

$$d_1^*=\frac{\ln\frac{X}{K}-\ln P(r,t;T)+\frac{1}{2}\int_t^T\widehat{\sigma}^2(\tau)d\tau}{\sqrt{\int_t^T\widehat{\sigma}^2(\tau)d\tau}}\tag{20}$$

$$d_2^* = d_1^* - \sqrt{\int_t^T \widehat{\sigma}^2(\tau)d\tau} \tag{21}$$

This formula was first proposed by Merton in 1973, when he was not received the random model of short-term interest rate r, but directly starting from the zero-coupon P_t, assuming P_t to meet the geometric Brownian motion, under the martingale measure, it is described by the following Stochastic differential equations,

$$\frac{dP_t}{P} = rdt + \sigma_P dB_t^1 \tag{22}$$

where $\{B_t^1 : t \geq 0\}$ is standard Brown motion, σ_P is the zero-coupons (bonds) volatility. Thus, European call option pricing formula is

$$V(X,P,t) = XN(d_1^*) - KPN(d_2^*) \tag{23}$$

which

$$\begin{aligned} d_1^* &= \frac{\ln\frac{X}{K} - \ln P + \frac{1}{2}\int_t^T \widehat{\sigma}^2(\tau)d\tau}{\sqrt{\int_t^T \widehat{\sigma}^2(\tau)d\tau}} \\ d_2^* &= d_1^* - \sqrt{\int_t^T \widehat{\sigma}^2(\tau)d\tau} \\ \widehat{\sigma}^2(\tau) &= \sigma_1^2 + \sigma_P^2 - 2\rho\sigma_1\sigma_P \end{aligned} \tag{24}$$

3. CONCLUSION

Compared with the standard European option pricing formula and European option under stochastic interest rate, there are only two differences: one is zero-coupon replaced by $e^{-r(T-t)}$; another is that using $\widehat{\sigma}$ instead of stock price volatility $\sigma_1\mathbb{F}$. Except that the pricing formulas is exactly the same form.

Analysis from the actual markets, $\sigma_P \ll \sigma_1$ means zero-coupons (bonds) volatility is far smaller than stock markets volatility, but in general $\sigma_P = \sigma_P(t)$, $\sigma_P(t)$ is monotonic decreasing, and $\lim_{t\to T} \sigma_P(t) = 0$. $\widehat{\sigma}$ and σ_1 in fact has the minor difference. Because in general, stock prices and bond prices are positively correlated, then $\rho > 0$. Therefore, if $\sigma_P < 2\rho\sigma_1$, by Equation (24) we know $\widehat{\sigma} < \sigma_1$ holds. Therefore under stochastic interest rates, the price of an option but have slightly decreased.

If a short-term interest rate model is given, only for the Vasicek model and Hull-White model [14], European option pricing formula has a simple form of the Merton formula (23). For C-I-R model, the corresponding zero-coupon stochastic model, the fluctuations in the rate of entry also including $\sqrt{r_t}$, so it can not write (22), so by pricing unit conversion of lower dimension than Number of purposes, such as lost that possible style of (24).

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