One Conclusion on the Essential Singularity of Analytic Function

ZENG Lijiang\textsuperscript{1,*}

\textsuperscript{1}Research Centre of Area Economy, Zunyi Normal College, Zunyi 563099, GuiZhou, China
\textsuperscript{*}Corresponding author.
Address: Research Centre of Area Economy, Zunyi Normal College, No: 830, Shanghai Road, Zunyi 563099, Guizhou Province, China.

Supported by Natural Science Foundation (13116339) of China; Natural Science Foundation ([2012]2069) of Science and Technology Department of Guizhou; Natural Science Foundation ([2012]712) of Education Department of Guizhou; Science Research item (2010028) of Zunyi Normal College.

Received December 30, 2011; accepted March 31, 2012

Abstract
In the article, the isolated singularity, removable singularity, zero, pole, essential singularity and other concepts and properties were used; Two lemmas on the isolated singularity were proved first, and to use these lemmas, one property of essential singularity was proved then.

Key words
Analytic function; Isolated singularity; Essential singularity; Laurent expansion

1. INTRODUCTION
The deep research shows that theory of complex function has extensive application in physics, engineering, and other subject\textsuperscript{[1–6]}.

As the important research content, in the article, the isolated singularity, removable singularity, zero, pole, essential singularity and other concepts and properties\textsuperscript{[7–8]} were used; Two lemmas on the isolated singularity were proved first, and to use these lemmas, one property of essential singularity was proved then.

2. THE PROOF OF LEMMAS; PREPARATION

Lemma 1 Let \( f \) be analytic on a region \( A \) and have an isolated singularity\textsuperscript{[1–6]} at \( z_0 \).

(i) \( z_0 \) is a removable singularity\textsuperscript{[9–10]} iff any one of the following conditions holds: (1) \( f \) is bounded in a deleted neighborhood\textsuperscript{[11–14]} of \( z_0 \); (2) \( \lim_{z \to z_0} f(z) \) exists; or (3) \( \lim_{z \to z_0} (z - z_0)f(z) = 0 \). (Note that it is not immediately evident that these three conditions are equivalent but the assertion is that they are and that each is equivalent to the condition that \( f \) has a removable singularity.

(ii)\( z_0 \) is a simple pole iff \( \lim_{z \to z_0} (z - z_0)f(z) \) exists and is unequal to zero. This limit equals the residue of \( f \) at \( z_0 \).
(iii) $z_0$ is a pole of order $\leq k$ (or possibly a removable singular) iff any one of the following conditions holds: (1) There is a constant $M > 0$ and an integer $k \geq 1$ such that $f(z) \leq \frac{M}{|z-z_0|^k}$ in a deleted neighborhood of $z_0$; (2) $\lim \limits_{z \to z_0} it(z - z_0)^k f(z) = 0$; or (3) $\lim \limits_{z \to z_0} it(z - z_0)^{k+1} f(z)$ exists.

(iv) $z_0$ is a pole of order $k \geq 1$ iff there is an analytic function $\varphi$ defined on a neighborhood $U$ of $z_0$ such that $U \setminus \{z_0\} \subset A$ such that $\varphi(z_0) \neq 0$, and such that

$$f(z) = \frac{\varphi(z)}{(z-z_0)^k} \text{ for } z \in U, \; z \neq z_0.$$  

**Proof.** (i) If $z_0$ is a removable singularity, then in a deleted neighborhood of $z_0$ we have $f(z) = \sum \limits_{n=0}^{\infty} a_n(z-z_0)^n$. Since this series represents an analytic function in an undeleted neighborhood of $z_0$, obviously conditions (1), (2), and (3) hold. Conditions (1) and (2) each obviously implies condition (3), so it remains to be shown that (3) implies that $z_0$ is a removable singularity for $f$. We must prove that each $b_k$ in the Laurent expansion

$$f(z) = \sum \limits_{n=0}^{\infty} a_n(z-z_0)^n$$

has a pole of order $k$. Let $b_k$ be the $z_0$ be given. By condition (3) can choose $r>0$ with $r < 1$ such that $\gamma_r$ is a circle whose interior (except for $z_0$) lies in $A$. Let $\varepsilon > 0$ be given. Then we can choose $b_k$ such that $\varphi(\zeta) \neq 0$ in $A$.

Then

$$|b_k| \leq \frac{1}{2\pi} \int_{\gamma_r} |f(\zeta)||(z-z_0)^{k-1}d\zeta| \leq \frac{1}{2\pi} \int_{\gamma_r} |\zeta|d\zeta = \frac{\varepsilon}{2\pi} k-1 \leq \varepsilon$$

Thus $|b_k| \leq \varepsilon$. Since $\varepsilon$ was arbitrary, $b_k=0$. We shall use (iii) to prove (ii). so (iii) is proved next.

(iii) This proof follows immediately by applying (i) to the function $(z-z_0)^k f(z)$, which is analytic on $A$. (One can easily obtained the details of the proof) (ii) If $z_0$ is a simple pole, then in a deleted neighborhood of $z_0$ we have

$$f(z) = \frac{b_1}{z-z_0} + \sum \limits_{n=0}^{\infty} a_n(z-z_0)^n = \frac{b_1}{z-z_0} + h(z)$$

where $h$ is analytic at $z_0$ and where $b_1 \neq 0$ by the Laurent expansion. Hence

$$\lim \limits_{z \to z_0} it(z-z_0) f(z) = \lim \limits_{z \to z_0} it(b_1 + (z-z_0) h(z)) = b_1.$$  

On the other hand, suppose that $\lim \limits_{\gamma \to z_0} \frac{it(z-z_0)}{f(z)}$ exists and is unequal to zero. Thus $\lim \limits_{z \to z_0} it(z-z_0)^2 f(z) = 0$. By the result obtained in (iii), this says that

$$f(z) = \frac{b_1}{z-z_0} + \sum \limits_{n=0}^{\infty} a_n(z-z_0)^n = \frac{b_1}{z-z_0} + h(z)$$

for some constant $b_1$, and analytic function $h$. where $b_1$ may or may not be zero. But then $\lim \limits_{z \to z_0} f(z) = b_1 + (z-z_0) h(z)$, so $\lim \limits_{z \to z_0} it(z-z_0) f(z) = b_1$. Thus, in fact, $b_1 \neq 0$, and therefore $f$ has a simple pole at $z_0$.

(iv) $z_0$ is a pole of order $k \geq 1$ iff

$$f(z) = \frac{b_k}{(z-z_0)^k} + \frac{b_{k-1}}{(z-z_0)^{k-1}} + \cdots + \frac{b_1}{(z-z_0)} + \sum \limits_{n=0}^{\infty} a_n(z-z_0)^n$$

$$= \frac{1}{(z-z_0)^k} \left\{ b_k + b_{k-1}(z-z_0) + \cdots + b_1(z-z_0)^{k-1} + \sum \limits_{n=0}^{\infty} a_n(z-z_0)^{n+k} \right\}, \quad (b_k \neq 0)$$

(If $b_k \neq 0$). This expansion is valid in a deleted neighborhood of $z_0$. \varphi(z) = b_k + b_{k-1}(z-z_0) + \cdots + b_1(z-z_0)^{k-1} + \sum \limits_{n=0}^{\infty} a_n(z-z_0)^{n+k}$. Then $\varphi(z)$ is analytic in the corresponding undeleted neighborhood (since it is a convergent power series) and $\varphi(z_0) = b_k \neq 0$. Conversely, given such a $\varphi$, we can retrace these steps to show that
Theorem 1 Let $f$ be analytic on a region $A$ and let $z_0 \in A$. We say that $f$ has a zero of order $k$ at $z_0$ if $f(z_0) = 0, \ldots, f^{(k-1)}(z_0) = 0$, $f^{(k)}(z_0) \neq 0$.

From the Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

we see that $f$ has a zero of order $k$ if, in a neighborhood of $z_0$, we can write $f(z) = (z - z_0)^k g(z)$ where $g(z)$ is analytic at $z_0$ and $g(z_0) = \frac{f^{(k)}(z_0)}{k!} \neq 0$. Thus from Lemma 1(iv), let $\varphi(z) = g(z)^{-1}$ we get the following.

**Lemma 2** If $f$ is analytic in a neighborhood of $z_0$, then $f$ has a zero of order $k$ at $z_0$ if $\frac{1}{f(z)}$ has a pole of order $k$ at $z_0$. If $h$ is analytic and $h(z_0) \neq 0$, then $\frac{h(z)}{f(z)}$ also has a pole of order $k$.

Obviously, if $z_0$ is a zero of $f$ and $f$ is not identically equal to zero in a neighborhood of $z_0$, then $z_0$ has some finite order $k$. (Otherwise the Taylor series would be identically zero.)

**Definition 1** In practical problems we usually are dealing with a pole. It is not hard to show that if $f(z)$ has a pole (of finite order $k$) at $z_0$, then $\left| f(z) \right| \to \infty$ as $z \to z_0$. However, in case of an essential singularity, $|f|$ will not, in general, approach $\infty$, as $z \to z_0$. In fact, we have the following result.

### 3. FINAL THEOREM AND ITS PROOF

**Theorem 1** Let $f$ have an essential singularity at $z_0$ and let $U$ be any (arbitrarily small) deleted neighborhood of $z_0$. Then, for all $w \in C$, except perhaps one value, the equation $f(z) = w$ has infinitely many solutions $z$ in $U$.

Theorem 1 actually belongs in a more advanced course. However, we can easily prove a simple version.

**Theorem 2** Let $f$ have an essential singularity at $z_0$ and let $w \in C$. Then there exist $z_1, z_2, z_3, \ldots, z_n \to z_0$, such that $f(z) \to w$.

**Proof** If the assertion were false, there would be a deleted neighborhood $U$ of $z_0$ and an $\varepsilon > 0$ such that $|f(z) - w| \geq \varepsilon$ for all $z \in U$. Let $g(z) = \frac{1}{f(z) - w}$. Thus on $U$, $g$ is analytic, and since $g(z)$ is bounded on $U(|g(z)| \leq \varepsilon^{-1}) \in C$ is removable singularity by Lemma 1(i). Let $k$ be the order of the zero of $g$ at $z_0$ (set $k = 0$ if $g(z_0) = 0$). (The order must be finite because otherwise, as mentioned previously, by the Taylor Theorem, $g$ would be zero in a neighborhood of $z_0$, whereas $g$ is 0 nowhere on $U$.) Thus $f(z) = w + \frac{1}{g(z)}$ is either analytic (if $k=0$) or has a pole of order $k$ by Lemma 2. This conclusion contradicts our assumption that $f$ has an essential singularity.

### 4. ACKNOWLEDGMENTS

This work was supported by Natural Science Foundation (13116339) of China; Natural Science Foundation ([2012]2069) of Science and Technology Department of Guizhou; Natural Science Foundation ([2012]712) of Education Department of Guizhou; Science Research item (2010028) of Zunyi Normal College.

### REFERENCES


