Pseudo-Parallel Legendrian Submanifolds With Flat Normal Bundle of Sasakian Space Forms

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Abstract: Let $M^n$ be a Legendrian submanifold with flat normal bundle of a Sasakian space form $M^{2n+1}(c)$. Further, $M^n$ is said to be pseudo-parallel if its second fundamental form $h$ satisfies $R(X,Y) \cdot h = L(X \wedge Y \cdot h)$. In this article we shall prove that $M$ is semi-parallel or totally geodesic and if $M$ satisfies $L \neq \frac{c+3}{4}$ then it is minimal in case of $n \geq 2$. Moreover, we show that if $M^n$ is also a H-umbilical submanifold then either $M^n$ is $L = \frac{c+3}{4}$, or $n = 1$.

Key words: Legendrian submanifold; Minimal submanifold; H-umbilical submanifold; Pseudo-parallel submanifold; Sasakian space form
1. INTRODUCTION

Recall that an isometric immersion $f: M^n \to \widetilde{M}^{n+k}$ from an $n$-dimensional Riemannian manifold into $n + k$-dimensional Riemannian with metric $g$ is pseudo-parallel if its second fundamental form $h$ satisfies

$$\mathcal{R}(X,Y) \cdot h = LX \wedge Y \cdot h,$$

where $\mathcal{R}(X,Y)$ is the curvature operator with respect to the van der Waerden-Bortolotti connection $\nabla$ of $f$, $L$ is some suitable smooth function on $M$ and $X \wedge Y$ is an operator: $(X \wedge Y)Z = g(X,Z)Y - g(Y,Z)X$. So $M$ is also referred as a $L$-pseudo-parallel submanifold of $\widetilde{M}$. In particular, if $L \equiv 0$, $M$ is called a semi-parallel submanifold.

In fact, the definition of pseudo-parallel was introduced in [1],[2] as an natural extension of semi-parallel and as the extrinsic analogue of pseudo-symmetry in the sense of Deszcz [7], i.e., the curvature operator of a semi-Riemannian manifold $(M, g)$ satisfies

$$R(X,Y) \cdot R = LRX \wedge Y \cdot R,$$

for any $X, Y$ tangent to $M$, $LR$ being some real value function on $M$.

Recently, concerning the study of pseudo-parallel immersion there are many results (see [1,2,9–11]), where the ambient manifold $\widetilde{M}$ has constant sectional curvature. In particular, we observe that Chacón and Lobos [5] studied pseudo-parallel Lagrangian submanifolds in a complex space form, and gave several properties. Also, they proved a local classification of pseudo-parallel Lagrangian surfaces. Analogous to the Lagrangian submanifolds in a complex space form, we consider $M^n$ is a Legendrian submanifold in Sasakian space form. Such a submanifold has been deeply studied over the past of several decades. However, for the pseudo-parallel submanifolds in a Sasakian space form, there is only A.Yildiz etc’s result [14], where they considered pseudo-parallel C-totally real minimal submanifolds in a Sasakian space form $\widetilde{M}^{2n+1}(c)$ and showed it is totally geodesic if $Ln - \frac{1}{4}(n(c + 3) + c - 1) \geq 0$. In the present paper we consider the Legendrian submanifolds with flat normal bundle in Sasakian space forms, which satisfy pseudo-parallel condition (1.1).

In section 2 we introduce some necessary basic conceptions and give some properties. The section 3 is our main results.

2. BASIC CONCEPTS

2.1. Sasakian Space Form

Let $\widetilde{M}^{2n+1}$ be a $2n + 1$-dimensional Riemannian manifold. $\widetilde{M}$ is called an almost contact manifold if it is equipped with an almost contact structure $(\phi, \xi, \eta)$, where $\phi$ is a $(1,1)$-tensor field, $\xi$ a unit vector field, $\eta$ a one-form dual to $\xi$ satisfying

$$\phi^2 = -I + \eta \otimes \xi, \eta \circ \phi = 0, \phi \circ \xi = 0.$$

(2.2)
It is well-known that there exists a Riemannian metric $\tilde{g}$ such that
\begin{align}
\tilde{g}(\phi X, \phi Y) &= \tilde{g}(X, Y) - \eta(X)\eta(Y), \\
\tilde{g}(X, Y) &= -\tilde{g}(X, \phi Y), \quad \tilde{g}(X, \xi) = \eta(X),
\end{align}
where $X, Y \in \mathfrak{X}(\tilde{M})$. Moreover, if the almost contact structure $(\phi, \xi, \eta)$ is normal, i.e.
\[ (\tilde{\nabla}_X \phi) Y = \tilde{g}(X, Y)\xi - \eta(Y)X, \quad \tilde{\nabla}_X \xi = -\phi X, \]
for any vectors $X, Y$ on $\tilde{M}$, where $\tilde{\nabla}$ denotes the connection with respect to $\tilde{g}$, then $\tilde{M}$ is said to be a Sasakian manifold. For more details and background, see [4] and [13].

A plane of $T_p \tilde{M}$ at $p$ is called $\phi$-section if it is spanned by $X$ and $\phi X$, where $X$ is orthonormal to $\xi$. The curvature of $\phi$-section is called $\phi$-sectional curvature.

A $2n + 1$-Sasakian space form is defined as a $2n + 1$-Sasakian manifold with constant $\phi$-sectional curvature $c$ and is denoted by $\tilde{M}^{2n+1}(c)$. As examples of Sasakian space form, $\mathbb{R}^{2n+1}$ and $S^{2n+1}$ are equipped with Sasakian space form structures (more details in [3] and [13]). The curvature of a Sasakian space form $\tilde{M}^{2n+1}(c)$ is given by [13]
\[ \tilde{R}(X, Y)Z = \frac{c+3}{4}(\tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y) + \frac{c-1}{4}(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \tilde{g}(X, Z)\eta(Y)\xi - \tilde{g}(Y, Z)\eta(X)\xi + \tilde{g}(\phi Y, Z)\phi X - \tilde{g}(\phi X, Z)\phi Y - 2\tilde{g}(\phi X, Y)\phi Z), \]
for any $X, Y, Z \in T\tilde{M}$.

2.2. Pseudo-Parallel Legendrian Submanifolds

Let $M^n$ be an $n$-dimensional submanifold of a Sasakian space form $\tilde{M}^{2n+1}(c)$. If the one-form $\eta$ constrained in $M$ is zero, then we say $M$ is a Legendrian submanifold. It is well-known that for such a submanifold $\phi$ maps any tangent vector to $M$ at any $p \in M$ into the normal vector space $T^\perp_p M$, i.e. $\phi T_p M \subset T^\perp_p M$. Actually, a Legendrian submanifold is special a $C$-totally real submanifold (i.e. the unit vector field $\xi$ is orthonormal to $M$). Therefore we obtain from (2.3) and (2.4) that for any $X, Y \in TM$,
\[ \tilde{g}(\phi X, \phi Y) = g(X, Y), \quad \eta(X) = \tilde{g}(X, \xi) = 0, \]
where $g$ is the induced metric of $\tilde{g}$. As usual, $\nabla$ and $\nabla^\perp$ denote by the Lev-Civita connection and normal connection on $M$, respectively. Then
\[ \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \]
where $h$ is the second fundamental form. Similarly, the Weingarten formula is:
\[ \tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N, \]
where $N$ is an normal vector on $M$ and $A_N$ is the shape operator. The shape operator is related to the second fundamental form by

$$g(A_N X, Y) = \tilde{g}(h(X, Y), N) = g(X, A_N Y).$$ (2.6)

If $R$ and $R^\perp$ denote, respectively, the Riemannian curvature tensors corresponding to $\nabla$ and $\nabla^\perp$, then the basic Gauss equation and Ricci equation are:

$$\tilde{g}(\tilde{R}(X, Y)Z, W) = g(R(X, Y)Z, W) + \tilde{g}(h(X, Z), h(Y, W)) - \tilde{g}(h(X, W), h(Y, Z)),$$

$$\tilde{g}(\tilde{R}(X, Y)N, V) = \tilde{g}(R^\perp(X, Y)N, V) - g([A_N, A_V]X, Y), \forall N, V \in T^\perp M.$$

The Codazzi equation:

$$(\tilde{R}(X, Y)Z) \perp = (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z).$$

Here $\nabla = \nabla \oplus \nabla^\perp$ stands for the Van der Waerden-Bortoloti connection, given by

$$(\nabla_X h)(Y, Z) = \nabla^\perp_X h(Y, Z) - h(\nabla^\perp_X Y, Z) - h(Y, \nabla_X Z).$$

Moreover, the following facts are well-known:

**Lemma 2.1** (see [8]). For a Legendrian submanifold, the following equations hold:

$$A_{\phi X} Y = A_{\phi Y} X,$$ (2.7)

$$A_{\phi X} Y = -\phi h(X, Y) = A_{\phi Y} X, \quad A_\xi = 0,$$ (2.8)

$$\tilde{g}(h(X, Y), \phi Z) = \tilde{g}(h(X, Z), \phi Y).$$ (2.9)

Therefore, it reduces from (2.2) and (2.8) that

$$\phi A_{\phi X} Y = h(X, Y) = \phi A_{\phi Y} X.$$ (2.10)

Moreover, using (2.10) and (2.4), from the Gauss equation we get

$$\tilde{R}(X, Y) = R(X, Y) - [A_{\phi X}, A_{\phi Y}].$$ (2.11)

For any vector fields $Z, W$ on $M$, the curvature operator $\tilde{R}(X, Y)$ with respect to $\nabla$ and $X \wedge Y$ can be extended as derivations of tensor fields in usual way. Therefore

$$(\tilde{R}(X, Y) \cdot h)(Z, W) = R^\perp(X, Y)(h(Z, W))$$

$$- h(R(X, Y)Z, W) - h(Z, R(X, Y)W),$$ (2.12)

$$(- X \wedge Y \cdot h)(Z, W) = - h((- X \wedge Y)Z, W) - h(Z, (- X \wedge Y)W)$$

$$= - g(Y, Z)h(X, W) + g(X, Z)h(Y, W)$$

$$- g(Y, W)h(X, Z) + g(X, W)h(Y, Z).$$ (2.13)

By (2.10),(2.12) and (2.13), if the normal bundle is flat, i.e. $R^\perp = 0$, then (1.1) becomes

$$- A_{\phi W} R(X, Y)Z - A_{\phi Z} R(X, Y)W$$

$$= L\{ - g(Y, Z)A_{\phi X} W + g(X, Z)A_{\phi Y} W$$

$$- g(Y, W)A_{\phi X} Z + g(X, W)A_{\phi Y} Z\}. $$ (2.14)
That is, a Legendrian submanifold $M^n$ with flat normal bundle of $\tilde{M}^{2n+1}(c)$ is pseudo-parallel if only if the Equation (2.14) is satisfied. In particular, if $L \equiv 0$, then $M$ is said to be semi-parallel. It is obvious that a totally geodesic submanifold is semi-parallel.

The following two propositions are the analogous conclusions to [5, Prop.3.1, Prop.3.2] in case of pseudo-parallel Legendrian submanifolds, respectively.

**Proposition 2.2.** Let $M^n$ be a pseudo-parallel Legendrian submanifold of Sasakian space form $\tilde{M}^{2n+1}$. If there exists another function $L'$ satisfies (1.1), then $L = L'$ at least $M - V$, where $V = \{p \in M|h_p = 0\}$.

**Proof.** If two functions $L$ and $L'$ satisfy (1.1), we have

$$(L - L')X \wedge Y \cdot h = 0.$$  

Choosing an orthonormal basis $\{e_1, \cdots, e_n\}$ of $T_pM$, $p \in M$, we get

$$(L - L') [e_i \wedge e_j \cdot h][e_k, e_l] = (L - L') [-h((e_i \wedge e_j)e_k, e_l) - h(e_k, (e_i \wedge e_j)e_l)]$$  

$$= (L - L') \{-g(e_j, e_k)h(e_i, e_l) + g(e_i, e_k)h(e_j, e_l)$$  

$$- g(e_j, e_l)h(e_i, e_k) + g(e_i, e_l)h(e_j, e_k)\}$$  

$$= (L - L') \{-\delta_{jk}h(e_i, e_l) + \delta_{ik}h(e_j, e_l) - \delta_{jl}h(e_i, e_k) + \delta_{il}h(e_j, e_k)\} = 0.$$  

For $i = k \neq j = l$, we have

$$(L - L')(h(e_i, e_i) - h(e_j, e_j)) = 0.$$  

For $i = k = l \neq j$, we have

$$(L - L') h(e_i, e_j) = 0.$$  

If $L(p) \neq L'(p)$ for $p \in M$, then

$$h(e_i, e_j) = 0, h(e_i, e_i) = h(e_j, e_j) \text{ for } \forall i \neq j.$$  

On the other hand, since when $i \neq j$,

$$g(h(e_i, e_i), \phi e_j) = g(h(e_i, e_j), \phi e_i) = 0,$$

$$g(h(e_i, e_i), \phi e_i) = g(h(e_j, e_j), \phi e_i) = g(h(e_i, e_j), \phi e_i) = 0,$$

$$g(h(e_i, e_i), \xi) = 0,$$

thus we obtain $g(h(e_i, e_i), N) = 0$, $i = 1, \cdots, n$, $\forall N \in T^\perp M$ since $\{\phi e_1, \cdots, \phi e_n, \xi\}$ is a basis of $T^\perp M$ for a Legendrian submanifold, that is, $h \equiv 0$. Consequently,

$$\{p \in M|L(p) \neq L'(p)\} \subseteq V.$$  

It leads to the proposition.
Proposition 2.3. Let $M^n$ be a pseudo-parallel Legendrian submanifold of Sasakian space form $\tilde{M}^{2n+1}(c)$ with flat normal bundle, then for any vector fields $X, Y \in TM$ we have

\[
R(X, Y)\phi H = L\{g(\phi H, X)Y - g(\phi H, Y)X\}.
\]

Here $H = \frac{1}{n}trh$ is the mean curvature vector.

Proof. Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame of $M$, $Z$ unit vector field of $T_p M$ for $p \in M$. For any vector field $U$ on $M$, using (2.6), we obtain from (2.14)

\[
g(R(X, Y)Z, A_{\phi W}U) + g(R(X, Y)W, A_{\phi Z}U)
= L\{g(Y, Z)g(A_{\phi X}W, U) - g(X, Z)g(A_{\phi Y}W, U)
+ g(Y, W)g(A_{\phi X}Z, U) - g(X, W)g(A_{\phi Y}Z, U)\}.
\]

Taking $W = U = e_j$ in (2.15), we obtain

\[
g(R(X, Y)Z, A_{\phi e_j}e_j) + g(R(X, Y)e_j, A_{\phi Z}e_j)
= L\{g(Y, Z)g(A_{\phi e_j}e_j, e_j) + g(Y, e_j)g(A_{\phi Z}e_j, e_j)
- g(X, Z)g(A_{\phi Y}e_j, e_j) - g(X, e_j)g(A_{\phi Y}Z, e_j)\}.
\]

Assume now that $\{\lambda_j\}_{j=1}^n$ are the eigenvalues of $A_{\phi Z}$ corresponding to frame $\{e_j\}_{j=1}^n$. Using (2.6) and (2.7), we get

\[
-g(R(X, Y)A_{\phi e_j}e_j, Z) + \lambda_j g(R(X, Y)e_j, e_j)
= L\{g(Y, Z)g(A_{\phi e_j}e_j, X) + g(Y, e_j)g(A_{\phi Z}e_j, X)
- g(X, Z)g(A_{\phi e_j}e_j, Y) - g(X, e_j)g(A_{\phi Z}e_j, Y)\}
= L\{g(Y, Z)g(A_{\phi e_j}e_j, X) + \lambda_j g(Y, e_j)g(e_j, X)
- g(X, Z)g(A_{\phi e_j}e_j, Y) - \lambda_j g(X, e_j)g(e_j, Y)\}.
\]

i.e.

\[
-g(R(X, Y)A_{\phi e_j}e_j, Z) = L\{g(Y, Z)g(A_{\phi e_j}e_j, X) - g(X, Z)g(A_{\phi e_j}e_j, Y)\}.
\]

Therefore

\[
g(R(X, Y)\phi H, Z) = -\frac{1}{n}\sum_{j=1}^n g(R(X, Y)A_{\phi e_j}e_j, Z)
= L\{g(Y, Z)g(\phi H, X) - g(X, Z)g(\phi H, Y)\}.
\]

It completes the proof of proposition.

3. MAIN RESULTS

Theorem 3.1. Let $M^n$ be a Legendrian submanifold of Sasakian space form $\tilde{M}^{2n+1}(c)(c \leq 1)$ with flat normal bundle, then $M^n$ is pseudo-parallel if and only if it is semi-parallel or totally geodesic.
Proof. By the curvature tensor (2.5) of $\widetilde{M}$, for any $X, Y \in TM$ we have

$$\widetilde{R}(X, Y)\phi H = \frac{c+3}{4}(g(Y, \phi H)X - g(X, \phi H)Y).$$

Since $R^\perp = 0$, the Ricci equation reduces to $[A_{\phi X}, A_{\phi Y}] = 0$, which implies $\widetilde{R}(X, Y)\phi H = R(X, Y)\phi H$ in view of (2.11). So, by Proposition 2.3 we have

$$\left(L + \frac{c+3}{4}\right)(g(\phi H, Y)X - g(\phi H, X)Y) = 0.$$ 

It yields $L = -\frac{c+3}{4}$ or $H = 0$.

When $L = -\frac{c+3}{4}$, if $c = -3$, it means that $L = 0$. If $c \neq -3$, i.e. $L \neq 0$, then it is easy to get

$$-g(Y, Z)A_{\phi X}W + g(X, Z)A_{\phi Y}W - g(Y, W)A_{\phi X}Z + g(X, W)A_{\phi Y}Z = 0 \quad (3.16)$$

due to (2.14). Thus, from (3.16) and making use of analogous argument to Proposition 2.2, we have $h = 0$, that is, $M$ is totally geodesic.

Next we assume that $L \neq -\frac{c+3}{4}$, then $H = 0$. By (2.5), for any vector $X, Y, Z$ tangent to $M$,

$$\widetilde{R}(X, Y)Z = \frac{c+3}{4}(g(Y, Z)X - g(X, Z)Y). \quad (3.17)$$

By making use of (2.14) and (3.17), we have

$$\left(\frac{c+3}{4} - L\right)\left\{-g(Y, Z)A_{\phi X}W + g(X, Z)A_{\phi Y}W - g(Y, W)A_{\phi X}Z + g(X, W)A_{\phi Y}Z = 0\right\} = 0.$$ \quad (3.18)

If we set $X = W = e_i$, and sum over $i = 1, \cdots , n$, using that $H = 0$, we obtain $L = \frac{c+3}{4}$ or $A_{\phi Y}Z = 0$ for any $Y$ and $Z$. The second case means that $M$ is totally geodesic. Assume in following $L = \frac{c+3}{4}$. Notice that in [14] A. Yildiz et al gave an necessary condition for a minimal pseudo-parallel C-totally real submanifold to be totally geodesic is $L n - \frac{1}{4}(n(c+3) + c - 1) \geq 0$. Therefore, in this case, $M^n$ is also totally geodesic.

Conversely, if $M$ is semi-parallel or totally geodesic, obviously it is trivial pseudo-parallel.

For a constant curvature manifold if its normal bundle is flat, Cartan, E. proved the following well-known fact (see [6]):

**Lemma 3.2 (Cartan, E).** Let $M^n$ be a submanifold of constant curvature space $\widetilde{M}^n+k(c), \{\xi_\alpha\}$ local orthogonal normal vector fields, and $\{h_\alpha\}$ the second fundamental forms corresponding to $\{\xi_\alpha\}$. Then in every point of $M$, all the $H_\alpha$ are mutually diagonalizable if and only if the normal bundle of $M$ is flat.

By Lemma 3.2, for any $p \in M$ there exists a local orthogonal frame $\{e_i\}$ of $M^n$ such that all the second fundamental form tensors are mutually diagonalizable, namely, for any unit normal vector field $N$,

$$A_N(e_i) = \lambda_i^N e_i,$$

where $\lambda_i^N$ are the principle curvatures of $M$ with respect to $N$. 

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Moreover, we have
\[
\text{Theorem 3.3. For } n \geq 2, \text{ let } M^n \text{ be a Legendrian submanifold of Sasakian space form } \tilde{M}^{2n+1}(c) \text{ with flat normal bundle. If } M^n \text{ is pseudo-parallel, then } L = \frac{c+3}{4} \text{ or } M \text{ is minimal.}
\]

**Proof.** For a Legendrian submanifold \( M \) we may choose an orthonormal basis of \( T^\perp_p M \) of the form \( \{e_{n+1} = \phi e_1, \ldots, e_{2n} = \phi e_n, e_{2n+1} = \xi \} \). For any \( i, j \in \{1, \ldots, n\} \), denote \( \lambda_i^{n+j} \) by the principle curvature with respect to normal vector field \( \phi e_j \), i.e.,
\[
A_{\phi e_j} e_i = \lambda_i^{n+j} e_i.
\]

Thus in this case the mean curvature vector can be written as
\[
H^{n+j} = \frac{1}{n} \sum_i \lambda_i^{n+j}.
\]

In view of (3.18), setting \( X = e_i, Y = e_j, Z = e_k, \) and \( W = e_l \), we have
\[
\left( \frac{c+3}{4} - L \right) \{ \delta_{il} A_{\phi e_j} e_k - \delta_{jk} A_{\phi e_j} e_l + \delta_{ik} A_{\phi e_j} e_l - \delta_{jl} A_{\phi e_j} e_k \} = 0,
\]
where \( g(e_i, e_j) = \delta_{ij} \) and \( 1 \leq i, j, k, l \leq n \). Using (3.19), we obtain
\[
\left( \frac{c+3}{4} - L \right) \{ \lambda_k^{n+j} \delta_{il} e_k - \lambda_l^{n+i} \delta_{jk} e_l + \lambda_l^{n+j} \delta_{ik} e_l - \lambda_k^{n+i} \delta_{jl} e_k \} = 0.
\]

Moreover, we have
\[
\left( \frac{c+3}{4} - L \right) \{ \lambda_k^{n+j} \delta_{il} \delta_{ks} - \lambda_l^{n+i} \delta_{jk} \delta_{ls} + \lambda_l^{n+j} \delta_{ik} \delta_{ls} - \lambda_k^{n+i} \delta_{jl} \delta_{ks} \} = 0.
\]

If we assume \( j = s, i = k \) in (3.20), it reduces to
\[
\left( \frac{c+3}{4} - L \right) \{ \lambda_l^{n+i} \delta_{il} - \lambda_l^{n+i} \delta_{il} + \lambda_l^{n+l} - \lambda_l^{n+i} \delta_{il} \} = 0.
\]

Further, by summing over \( i = 1, \ldots, n \), we have
\[
\left( \frac{c+3}{4} - L \right) \{ \sum_i \lambda_l^{n+i} \delta_{il} - \lambda_l^{n+l} + n \lambda_l^{n+l} - \sum_i \lambda_l^{n+i} \delta_{il} \} = 0.
\]

Because it follows from (2.7) that
\[
\lambda_i^{n+l} = g(A_{\phi e_1} e_i, e_i) = g(A_{\phi e_i} e_i, e_l) = \lambda_l^{n+i} \delta_{il},
\]

therefore Equation (3.21) implies
\[
\left( \frac{c+3}{4} - L \right) \lambda_l^{n+l} = 0.
\]

On the other hand, using (3.19), it follows from (2.15) that
\[
-\lambda_s^{n+l} R_{ijkl} - \lambda_s^{n+k} R_{ijkl} = L \{ -\delta_{jk} \delta_{ls} \lambda_l^{n+i} + \delta_{ik} \delta_{ls} \lambda_l^{n+j} - \delta_{jl} \delta_{ks} \lambda_k^{n+i} + \delta_{il} \delta_{ks} \lambda_k^{n+j} \}.
\]
Since
\[ R_{i j k s} = \frac{c + 3}{4} (\delta_{j k} \delta_{i s} - \delta_{i k} \delta_{j s}), \quad R_{i j l s} = \frac{c + 3}{4} (\delta_{j l} \delta_{i s} - \delta_{i l} \delta_{j s}), \quad (3.24) \]
by substituting (3.24) into (3.23), we get
\[ -\frac{c + 3}{4} \left\{ \lambda_{n+l}^{i} (\delta_{j k} \delta_{i s} - \delta_{i k} \delta_{j s}) + \lambda_{s}^{i+k} (\delta_{j l} \delta_{i s} - \delta_{i l} \delta_{j s}) \right\} = L \left\{ -\delta_{j k} \delta_{i s} \lambda_{l}^{j+i} + \delta_{i k} \delta_{l s} \lambda_{l}^{n+j} - \delta_{j l} \delta_{k s} \lambda_{k}^{n+j} + \delta_{i l} \delta_{k s} \lambda_{k}^{n+j} \right\}. \quad (3.25) \]
In the same way, putting \( i = k, j = s \) and summing over \( i = 1, \cdots, n \) and \( j = 1, \cdots, n \) in (3.25), respectively, we have
\[ \frac{c + 3}{4} H^{n+l} = L \lambda_{l}^{n+l}. \quad (3.26) \]
Combining (3.26) with (3.22), we concluded that if \( (L - \frac{c+3}{4}) H^{n+l} = 0 \). This completes the proof of theorem.

It is easy to show the following corollary from (3.26):

**Corollary 3.4.** For \( n \geq 2 \) and \( c \neq -3 \), let \( M^n \) be a Legendrian submanifold of Sasakian space form \( \tilde{M}^{2n+1}(c) \) with flat normal bundle. If \( M^n \) is semi-parallel then it is a minimal submanifold.

In [14, Corollary 6], the authors showed that for a minimal Legendrian submanifold \( M^n \) of Sasakian space form \( \tilde{M}^{2n+1}(c) \), if it is semi-parallel and satisfies \( n(c + 3) + c - 1 \leq 0 \), then it is totally geodesic. Thus by Corollary 3.4, we have the following corollary:

**Corollary 3.5.** For \( n \geq 2 \) and \( c < -3 \), let \( M^n \) be a Legendrian submanifold of Sasakian space form \( \tilde{M}^{2n+1}(c) \) with flat normal bundle. If \( M^n \) is semi-parallel then it is totally geodesic.

Note that Blair proved the following conclusion:

**Theorem 3.6 ([4]).** Let \( M^n \) be a minimal \( C \)-totally real submanifold of \( (2n + 1) \)-Sasakian space form \( \tilde{M}(c) \). Then the following are equivalent:

1) \( M^n \) is totally geodesic,
2) \( M^n \) is of constant curvature \( K = \frac{1}{4} (c + 3) \),
3) \( S = \frac{1}{4} (n - 1)(c + 3) \),
4) \( \kappa = \frac{1}{4} n(n - 1)(c + 3) \),

where \( S \) and \( \kappa \) are the Ricci curvature and scalar curvature of \( M \), respectively.

Since the normal bundle is flat, \( M^n \) is of constant curvature \( K = \frac{c+3}{4} \), in view of Corollary 3.4, we have
Corollary 3.7. Let $M^n$ be a Legendrian submanifold of Sasakian space form $\tilde{M}^{2n+1}(c)$ ($c \neq -3$) with flat normal bundle. If $M^n$ is semi-parallel the following conclusions are equivalent:

1) $M^n$ is totally geodesic,
2) $S = \frac{1}{4}(n-1)(c+3)$,
3) $\kappa = \frac{1}{4}n(n-1)(c+3)$.

Now recall that an non-totally geodesic Legendrian $H$-umbilical submanifold $M^n$ of Sasakian manifold $\tilde{M}^{2n+1}$ is a Legendrian submanifold and its second fundamental form satisfies the following forms:

$$
\begin{align*}
    h(e_1, e_1) &= \lambda \phi e_1, \\
    h(e_1, e_j) &= \mu \phi e_j, \\
    h(e_j, e_k) &= 0, \quad 2 \leq j \neq k \leq n,
\end{align*}
$$

for some suitable functions $\lambda$ and $\mu$ with respect to some suitable orthonormal local frame field $\{e_i\}$ of $M$ [12].

Theorem 3.8. Assume that $M^n$ is a Legendrian $H$-umbilical submanifold of Sasakian space form $\tilde{M}^{2n+1}(c)$ with flat normal bundle connection. If $M^n$ is pseudo-parallel then either $L = \frac{c+3}{4}$, or $n = 1$.

Proof. We consider $\{e_1, \cdots, e_n\}$ as (3.27), then from (3.18) we get

$$
\left( \frac{c + 3}{4} - L \right) \{ \delta_{il} A_{\phi, e_j} e_k - \delta_{jk} A_{\phi, e_i} e_l + \delta_{ik} A_{\phi, e_j} e_l - \delta_{jl} A_{\phi, e_i} e_k \} = 0. \tag{3.28}
$$

Assume that $j = 1$ and $i = k$ in (3.28), a straightforward calculation implies

$$
\left( \frac{c + 3}{4} - L \right) \{ (n-1)(\lambda - \mu) e_1 + n\mu \sum_{l=2}^{n} e_l \} = 0. \tag{3.29}
$$

If $L \neq \frac{c+3}{4}$ then the above equation implies that $\mu = 0$ and $(n-1)(\lambda - \mu) = 0$ since $\{e_i\}$ is orthonormal, that is, $n = 1$.

REFERENCES