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The Study of Partial Connective Stability for Singular Linear Large-Scale Interconnected Systems

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Abstract: The connective stability is an important content among the large-scale interconnected systems. The many results recently have been given for the normal systems. In this paper, we shall discuss the connective stability of singular linear large-scale dynamical systems by means of singular Lyapunovf equation and vector Lyapunovf function. We give a simple form of a stable domain of connective parameters.
Key words: Singular system; Large-scale systems; Stability; Connective stability; Vector Lyapunovf function

1. INTRODUCTION

With the development of modern control theory and the permeation into other application area such as aviation, spaceflight, energy, power, oil, chemical industry and communication and so forth, many researchers discover a kind of dynamical system which possesses more widely representative in form, that is singular system, which is of the following form.

$$I\dot{X}(t) = f(X(t),t)$$

Where I is a $n \times n$ matrix, it is usually singular. $X(t) \in R^n$ is a $n-$vector, f is a $n-$vector function. The concept of the singular system is first proposed in the 1970s, within short thirty years, it has become an independent branch of the modern control theory. Especially later decade more and more scholars are interested in studying of the singular system theory, and many important results have been obtained as (ZHANG, 1997; CHEN & LIU, 1998; YANG & ZHANG, 2004; Campbell, 1992). There exist singular systems models in many area of the social production, known Dynamic Leontief Input Output Model, Hopfield neural network model, multi-robot Subjective Coordinate Work Dynamic

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Model and so on are all singular systems. It is far-reaching practical significance to study singular system theory.

The connective stability is an important content among the large-scale dynamical interconnected systems. The many results recently have been given for the normal systems (SUN & CHEN, 2003; SUN & PENG, 2008; CHEN, 1998; LIU, 1989). In this paper by means of the singular Lyapunovf equation and vector Lyapunovf function we discuss the connective stability of linear singular large-scale system and give a connective parameters stable domain. An example to illustrating its efficiency is given.

2. BASIC CONCEPTION

We consider singular linear time-invariant large-scale dynamical system described by equations of the form

$$I_i \dot{X}_i(t) = A_{ii}X_i(t) + \sum_{j=1, j\neq i}^{r} e_{ij}A_{ij}X_j(t) \quad (1) \qquad i = 1,2,\cdots,r$$

Where I_i and A_{ii} are all $n_i \times n_i$ constant matrices, A_{ij} is a constant $n_i \times n_j$ matrix. $X_i \in R^{n_i} (i, j = 1,2,\cdots,r)$. We denote $n = \sum_{i=1}^{r} n_i$, $X^T = (X_1^T, X_2^T, \cdots, X_r^T) \in R^n$, $I = Block\text{-}diag(I_1, I_2, \cdots, I_r)$, and we assume that I is a singular matrix , and we assume for any initial time $t_0 \in R^+$ and any initial state $x_0 = x(t_0)$, there is one (and only one) solution $x = x(t; t_0, x_0)$ of the system (1).

A matrix $E = (e_{ij})_{r\times r}$ is called an interconnection matrix associated with $\bar{E}$, if $\bar{e}_{ij} = 0$ imply $e_{ij} = 0$ and if $\bar{e}_{ij} = 1$ imply $e_{ij} = 1$ or $e_{ij} = 0$. The elements $\bar{e}_{ij}$ of the fundamental interconnection matrix $\bar{E}$ are defined as

$$\bar{e}_{ij} = \begin{cases} 1, & X_j \ can \ \ act \ \ on \ \ X_i \\ 0, & X_j \ cann't \ \ act \ \ on \ \ X_i \end{cases}$$

It is denoted as $E \in \bar{E}$ (LIU, 1989).

The isolated subsystems of (1) are

$$I_i \dot{X}_i(t) = A_{ii}X_i(t), i = 1,2,\cdots,r. \qquad (2)$$

The system (2) is called regular (YANG, ZHANG, 2004), if there exists a constant s_0 for every $i (i = 1,2,\cdots,r)$, such that

$$\det(s_0 I_i - A_{ii}) \neq 0$$

The system (2) is called impulsive-free (YANG, ZHANG, 2004), if for any $s \in C$ (C domain of complex number), it always holds that

$$\deg \det(sI_i - A_{ii}) = rank(I_i), (i = 1,2,\cdots,r).$$

3. MAIN RESULTS

We always assume that for every isolated subsystem (2) is regular and impulsive-free.

If there exists an positive definite matrix B_i, satisfying the following singular Lyapunovf equation

$$A_{ii}^T B_i I_i + I_i^T B_i A_{ii} = -I_i^T I_i \qquad (3)$$

We take singular Lyapunovf function as follows

$$V_i[I_i X_i(t)] = [I_i X_i(t)]^T B_i [I_i X_i(t)] \qquad (4)$$

Then we get

$$\lambda_m(B_i)\left\|I_i X_i(t)\right\|^2 \le V_i[I_i X_i(t)] \le \lambda_M(B_i)\left\|I_i X_i(t)\right\|^2 \qquad (5)$$

Where $\|\cdot\|$ represents Euclidean norm, $\lambda_m(B)$ and $\lambda_M(B)$ represent respectively minimum and maximum eigenvalue of the positive definite matrix B. And the total derivative of $V_i[I_i X_i(t)]$ with respect to t along solutions of the systems (2) is

$$\dot{V}i[I_i X_i(t)]_{(2)} = -[I_i X_i]^T [I_i X_i] \qquad (6)$$

Lemma (CHEN & LIU, 1998): For any a pair of $X, Y \in R^n$, a positive definite matrix B and a constant number $\mu > 0$, the following inequality holds

$$X^T BY + Y^T BX \le \mu X^T BX + \mu^{-1} Y^T BY \qquad (7)$$

Definition (CHEN, 1998): $\bar{E}$ is the fundamental interconnection matrix of the system (1). The system (1) is called **connectively asymptotically stable,** if the equilibrium state $X^* = 0$ of the system (1) is always asymptotically stable for any $E \in \bar{E}$. Otherwise, the system (1) is called **connectively unstable.**

Theorem 1: The equilibrium state $X^* = 0$ of the system (1) is connectively asymptotically stable if the following conditions are satisfied

(ⅰ) ***We assume that every isolated subsystem (2) is asymptotically stable, in which there exists a positive definite matrix B_i, such that it satisfies (5) and (6);***

(ⅱ) ***for singular large-scale dynamical systems (1), there are positive numbers δ_{ij}, such that***

$$\left\|A_{ij} X_j(t)\right\| \le \delta_{ij}\left\|I_j X_j(t)\right\| \quad (i, j = 1,2,\cdots,r, j \ne i) \qquad \textbf{(8)}$$

(ⅲ) ***all successively principal minors C_k $(k = 1,2,\cdots,r)$ of the $r \times r$ matrix $C = (c_{ij})_{r\times r}$ satisfy***

$$(-1)^k C_k = (-1)^k \begin{vmatrix} c_{11} & \cdots & c_{1k} \\ \vdots & & \vdots \\ c_{k1} & \cdots & c_{kk} \end{vmatrix} > 0 \qquad \textbf{(9)}$$

where

$$c_{ij} = \begin{cases} \dfrac{-1+\lambda_M(B_i)}{\lambda_M(B_i)}, & i=j \\ \dfrac{(r-1)\bar{e}_{ij}\delta_{ij}^{\ 2}\lambda_M(B_i)}{\lambda_m(B_j)}, & i \neq j \end{cases} \qquad \textbf{(10)}$$

$$i, j = 1,2,\cdots,r$$

Proof: Let $Y_i = \left\| I_i X_i(t) \right\|$, $(i=1,2,\cdots r,)$ for any $E=(e_{ij})_{r\times r} \in \overline{E}$, taking singular vector Lyapunovf function as follows

$$V[IX(t)] = \{V_1[I_1X_1(t)], V_2[I_2X_2(t)], \cdots, V_r[I_rX_r(t)]\}^T$$

The total derivative of $V_i[I_iX_i(t)]$ with respect to t along solutions of the systems (1) is

$$\dot{V}i[I_iX_i(t)]_{(1)}$$

$$=[I_i\dot{X}i(t)]^T B_i[I_iX_i(t)] + [I_iX_i(t)]^T B_i[I_i\dot{X}i(t)]$$

$$=[A_{ii}X_i(t) + \sum_{j=1,j\neq i}^{r} e_{ij}A_{ij}X_j(t)]^T B_i[I_iX_i(t)]$$

$$+[I_iX_i(t)]^T B_i[A_{ii}X_i(t) + \sum_{j=1,j\neq i}^{r} e_{ij}A_{ij}X_j(t)]$$

$$=[X_i(t)]^T (A_{ii}^T B_i I_i + I_i^T B_i A_{ii}) X_i(t)$$

$$+2[\sum_{j=1,j\neq i}^{r} e_{ij}A_{ij}X_j(t)]^T B_i[I_iX_i(t)]$$

By (7), we get

$$\dot{V}i[I_iX_i(t)]_{(1)}$$

$$\leq \dot{V}_i[I_iX_i(t)]_{(2)} + [I_iX_i(t)]^T B_i[I_iX_i(t)]$$

$$+[\sum_{j=1,j\neq i}^{r} e_{ij}A_{ij}X_j(t)]^T B_i[\sum_{j=1,j\neq i}^{r} e_{ij}A_{ij}X_j(t)]$$

$$\leq -\left\|I_iX_i(t)\right\|^2 + \lambda_M(B_i)\left\|I_iX_i(t)\right\|^2$$

$$+\lambda_M(B_i)\|\sum_{j=1,j\neq i}^{r} e_{ij}A_{ij}X_j(t)\|^2$$

$$\leq -Y_i^2+\lambda_M(B_i)Y_i^2+\lambda_M(B_i)(\sum_{j=1,j\neq i}^{r}\bar{e}_{ij}\delta_{ij}Y_j)^2$$

$$\leq [-1+\lambda_M(B_i)]Y_i^2+\lambda_M(B_i)(r-1)\sum_{j=1,j\neq i}^{r}\bar{e}_{ij}\delta_{ij}^2Y_j^2$$

By the hypothesis (iii) of the theorem 1, it follows

From $[-1+\lambda_M(B_i)]<0$, and by the (5), we have

$$\dot{V}i[I_iX_i(t)]\leq\frac{-1+\lambda_M(B_i)}{\lambda_M(B_i)}V_i[I_iX_i(t)]$$

$$+\sum_{j=1,j\neq i}^{r}\frac{(r-1)\bar{e}_{ij}\delta_{ij}{}^2\lambda_M(B_i)}{\lambda_m(B_j)}V_j[I_jX_j(t)]$$

that is

$$\dot{V}_i[I_iX_i(t)]\leq\sum_{j=1}^{r}c_{ij}V_j[I_jX_j(t)](i=1,2,\cdots,r)$$

The total derivative of $V[IX(t)]$ with respect to t along solutions of the systems (1) holds following inequality

$$\dot{V}[IX(t)]_{(1)}\leq CV[IX(t)]$$

Since matrix C satisfies the hypothesis (iii) of theorem 1, all eigenvalues of matrix C have negative real parts . It follows from the Comparison principle that

$V[IX(t)]\to 0\quad (t\to+\infty)$

Therefore

$V_i[I_iX_i(t)]\to 0\quad (t\to+\infty)\quad (i=1,2,\cdots,r)$

and

$I_iX_i(t)\to 0\quad (t\to+\infty)\quad (i=1,2,\cdots,r)\quad (11)$

Since every isolated subsystem (2) is regular and impulsive-free, so there exist nonsingular matrices P_i, Q_i, such that

$$P_i^{-1}I_iQ_i=\begin{pmatrix} I_i^{(1)} & 0\\ 0 & 0\end{pmatrix}\qquad (12)$$

and

$$P_i^{-1}A_{ii}Q_i=\begin{pmatrix}A_{ii}^{(1)} & 0\\ 0 & I_i^{(2)}\end{pmatrix} \qquad (13)$$

denote

$$Q_i^{-1}X_i(t)=\overline{X}_i(t)=\begin{pmatrix}\overline{x}_{i1}(t)\\ \overline{x}_{i2}(t)\end{pmatrix}\quad (i=1,2,\cdots,r)$$

where $I_i^{(1)}, I_i^{(2)}$ are respectively $r_i\times r_i$ and $(n_i-r_i)\times(n_i-r_i)$ unit matrices, $\overline{x}_{i1}(t)\in R^{r_i}$, $\overline{x}_{i2}(t)\in R^{n_i-r_i}$, $A_{ii}^{(1)}$ is a $r_i\times r_i$ constant matrix.

By the (11) and

$$P_i^{-1}I_iX_i(t)=\begin{pmatrix}I_i^{(1)} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\overline{x}_{i1}(t)\\ \overline{x}_{i2}(t)\end{pmatrix}=\begin{pmatrix}\overline{x}_{i1}(t)\\ 0\end{pmatrix}$$

We get

$$\overline{x}_{i1}(t)\to 0\quad (t\to+\infty)\quad (i=1,2,\cdots,r)$$

By (8) and (11) we have

$$A_{ij}X_j(t)\to 0\quad (t\to+\infty)\quad (i,j=1,2,\cdots,r,j\neq i)$$

Since the system (1) can be written as

$$\begin{pmatrix}I_i^{(1)} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\dot{\overline{x}}_{i1}(t)\\ \dot{\overline{x}}_{i2}(t)\end{pmatrix}=\begin{pmatrix}A_{ii}^{(1)} & 0\\ 0 & I_i^{(2)}\end{pmatrix}\begin{pmatrix}\overline{x}_{i1}(t)\\ \overline{x}_{i2}(t)\end{pmatrix}+$$

$$+\sum_{j=1,j\neq i}^{r}e_{ij}\begin{pmatrix}A_{ij}^{(1)} & A_{ij}^{(2)}\\ A_{ij}^{(3)} & A_{ij}^{(4)}\end{pmatrix}\begin{pmatrix}\overline{x}_{j1}(t)\\ \overline{x}_{j2}(t)\end{pmatrix}$$

Where

$$P_i^{-1}A_{ij}Q_j=\begin{pmatrix}A_{ij}^{(1)} & A_{ij}^{(2)}\\ A_{ij}^{(3)} & A_{ij}^{(4)}\end{pmatrix}\quad (i,j=1,2,\cdots,r,i\neq j)$$

We get

$$\overline{x}_{i2}(t)=-\sum_{j=1,j\neq i}^{r}e_{ij}[A_{ij}^{(3)}\overline{x}_{j1}(t)+A_{ij}^{(4)}\overline{x}_{j2}(t)]$$

Then

$$\overline{x}_{i2}(t)\to 0\quad (t\to+\infty)\quad (i=1,2,\cdots r)$$

And

$$\overline{X}i(t) \to 0 \ \ (t \to +\infty) \ \ (i = 1,2,\cdots,r)$$

$$X_i(t) \to 0 \ \ (t \to +\infty) \ \ (i = 1,2,\cdots,r)$$

Finally we have

$$X(t) \to 0 \ \ (t \to +\infty)$$

That is, the equilibrium point $X^* = 0$ of (1) is connectively asymptotically stable.

Theorem 2: We assume that

(a) There exists a positive definite matrix B_i and a

Constant number $\mu_i > 0$ such that total derivative of* (4) *with respective to t along the solution of* (2) *satisfies

$$\dot{V}i[I_i X_i(t)]_{(2)} = \mu_i \left\| I_i X_i(t) \right\|^2 \qquad (14)$$

(b) For singular large-scale systems* (1)*, there exists positive constant numbers δ_{ij}, such that

$$\left\| A_{ij} X_j(t) \right\| \le \delta_{ij} \left\| I_j X_j(t) \right\| \qquad (15)$$

$$(i, j = 1,2,\cdots,r, i \ne j)$$

(c) All successively principal minors C_k of the $r \times r$ matrix $C = (c_{ij})_{r\times r}$ satisfy $C_k > 0 \ \ (k = 1,2,\cdots,r)$**,**

Then the equilibrium point $X^* = 0$ of the system (1) is connectively unstable, where

$$c_{ij} = \begin{cases} \dfrac{\mu_i - \lambda_M(B_i)}{\lambda_M(B_i)}, & i = j \\ -\dfrac{(r-1)\overline{e}_{ij}\delta_{ij}^{\ 2}\lambda_M(B_i)}{\lambda_m(B_j)}, & i \ne j \end{cases} \qquad (16)$$

Proof: Let $Y_i = \left\| I_i X_i(t) \right\|, \ i = 1,2,\cdots r,$ and for $E = (e_{ij}) = \overline{E}$, taking vector singular Lyapunovf function as follows

$$V[IX(k)] = \{V_1[I_1X_1(k)], V_2[I_2X_2(k)], \cdots, V_r[I_rX_r(k)]\}^T$$

The total derivative of $V_i[I_iX_i(t)]$ with respect to t along solutions of the systems (1) is

$$\dot{V}_i[I_iX_i(t)]_{(1)}$$

$$= [I_i \dot{X}i(t)]^T B_i [I_i X_i(t)] + [I_i X_i(t)]^T B_i [I_i \dot{X}i(t)]$$

By (7), we have

$$\dot{V}_i[I_iX_i(t)]_{(1)}$$

$$\geq \dot{V}_i[I_iX_i(t)]_{(2)} - [I_iX_i(t)]^T B_i[I_iX_i(t)]$$

$$-[\sum_{j=1,j\neq i}^{r} e_{ij}A_{ij}X_j(t)]^T B_i[\sum_{j=1,j\neq i}^{r} e_{ij}A_{ij}X_j(t)]$$

$$\geq \mu_i\left\|I_iX_i(t)\right\|^2 - \lambda_M(B_i)\left\|I_iX_i(t)\right\|^2$$

$$-\lambda_M(B_i)\|\sum_{j=1,j\neq i}^{r} e_{ij}A_{ij}X_j(t)\|^2$$

$$\geq \mu_iY_i^2 - \lambda_M(B_i)Y_i^2 - \lambda_M(B_i)(\sum_{j=1,j\neq i}^{r} \bar{e}_{ij}\delta_{ij}Y_j)^2$$

$$\geq [\mu_i - \lambda_M(B_i)]Y_i^2 - \lambda_M(B_i)(r-1)\sum_{j=1,j\neq i}^{r} \bar{e}_{ij}\delta_{ij}^2Y_j^2$$

By the condition (c) of the theorem 2, it follows form $[\mu_i - \lambda_M(B_i)] > 0$, and by the (5), we have

$$\dot{V}_i[I_iX_i(t)]_{(1)} \geq \frac{\mu_i - \lambda_M(B_i)}{\lambda_M(B_i)}V_i[I_iX_i(t)]$$

$$-\sum_{j=1,j\neq i}^{r}\frac{(r-1)\bar{e}_{ij}\delta_{ij}^2\lambda_M(B_i)}{\lambda_m(B_j)}V_j[I_jX_j(t)]$$

That is

$$\dot{V}_i[I_iX_i(t)]_{(1)} \geq \sum_{j=1}^{r}c_{ij}V_j[I_jX_j(t)] \qquad (i=1,2,\cdots,r)$$

The total derivative of $V[IX(t)]$ with respect to t along solutions of the systems (1) holds following inequality

$$\dot{V}[IX(t)]_{(1)} \geq CV[IX(t)]$$

By the condition (c), it follows from the Comparison principle that the equilibrium point $X^* = 0$ of (1) is connectively unstable.

4. FOR EXAMPLE

Consider following singular linear large-scale system which includes two subsystems with 5 order

$$\begin{cases} I_1\dot{X}_1 = A_{11}X_1 + e_{12}A_{12}X_2 \\ I_2\dot{X}_2 = e_{21}A_{21}X_1 + A_{22}X_2 \end{cases} \qquad (17)$$

Where

$$I_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_{11} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_{12} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 0 \end{pmatrix}, A_{21} = \begin{pmatrix} 1/2 & 2/3 \\ -2/3 & 1/2 \\ 0 & 0 \end{pmatrix}$$

We get positive definite matrices

$$B_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \; B_2 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 2/3 \end{pmatrix}$$

Taking $\overline{E} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, let $\delta_{12} = \delta_{21} = \delta$,

Then for $\forall E \in \overline{E}$, we have followed matrix

$$C = \begin{pmatrix} -1 & \delta/3 \\ \delta & -1/2 \end{pmatrix}$$

For $\delta < \sqrt{6}/2$, matrix C satisfies the condition (iii) of theorem 1, therefore, the equilibrium point $X^* = 0$ of the system (17) is connectively asymptotically stable.

5. CONCLUSION

The item e_{ij} in the paper is taken to be 0 or 1, in practice, the conclusions above are adapted to the condition $0 \le e_{ij}(t) \le 1$, that is to say, for any moment t , under the different intensity of connective among the sub-systems, the conclusion is still holds.

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