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w-(H,Ω) CONJUGATE DUALITY THEORY IN MULTIOBJECTIVE NONLINEAR OPTIMIZATION[1]

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Abstract: The duality in multiobjective optimization holds now a major position in the theory of multiobjective programming not only due to its mathematical elegance but also its economic implications. For weak efficiency instead of efficiency, this paper gives the definition and some fundamental properties of the weak supremum and infimum sets. Based on the weak supremum, the concepts, some properties and their relationships of w-(H,Ω) conjugate maps, w-(H,Ω)-subgradients, w- Γ_H^p (Ω)-regularitions of vector-valued point-to-set maps are provided, and a new duality theory in multiobjective nonlinear optimization------w-(H,Ω) Conjugate Duality Theory is established by means of the w-(H,Ω) conjugate maps. The concepts and their relations to the weak efficient solutions to the primal and dual problems of the w-(H,Ω)-Lagrangian map and weak saddle-point are developed. Finally, several special cases for H and Ω are discussed.

Key words: Conjugate duality theory, Multiobjective optimization, Weak efficiency

1. INTRODUCTION

In the duality theory of nonlinear programming problems, conjugate functions play important roles [1,2,8]. So, in order to discuss duality in multiobjective optimization problems, we need to introduce an extended notion of "conjugate" which fits well the multiobjective problems. Tanino and Sawaragi [9] have developed a duality theory in multiobjective convex programming problems under the Pareto optimality criterion. However, their duality theory is not reflexive in the sense analogous to [1 , 2]. The nonreflexivity of their duality seems to be caused by the fact that the objective functions in the primal problems are vector-valued, while those in the dual problems are set-valued. If we are concerned with the reflexivity of the duality theory, we will have to start from set-valued objective functions in the primal problems. Then a new problem comes about: what kinds of set-valued functions are suitable for objective functions ? Feng [4] extended the results of [9] to a more general framework, and provided a reflexive duality theory in multiobjective optimization based on efficiency. Based on weak efficiency rather than efficiency, Kawasaki [6,7] developed some interesting results by defining conjugate and subgradients via weak supremum. In this article, a new

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duality theory for weak efficiency is developed with the help of the weak supremum and the generalization of the conjugate relations discussed in [5]. In section 2, the definitions of a supremum set and a infimum set which are slightly different from Kawasaki's are given and some of their properties are provided. In section 3, we shall give the definitions of w-(H,Ω) conjugate maps and w-(H,Ω)-subgradients and w-Γ_H^p(Ω)-regulations of the point-to-set map, and discuss some their fundamental properties. The weak saddle-point problem is also discussed in this section. In section 4, the w-(H,Ω) conjugate duality theory is developed which states the relationships between the primal and dual problems. In section 5, the w-(H,Ω)-Lagrangian map is defined and its weak saddle-point problem is discussed. In section 6, several special cases for H and Ω are discussed.

Before going further, we introduce the following notations. Let R^p be the p-dimensional Euclidean space, and ε and -ε be the p-dimensional points consisting of +∞ and -∞, respectively, and $\underline{R}^p=R^p\cup\{\varepsilon,-\varepsilon\}$. $K=\{x\in R^p: x>0\}\cup\{0\}$. By K, we get two partial orders "$\le_K$" and "$<_K$" in R^p, that is, for any points $a,b\in R^p$, $a\le_K b$ if and only if $b-a\in K$, and $a<_K b$ if and only if $b-a\in int(K)=K\backslash\{0\}$. Furthermore, the partial orders can be extended naturally, i.e., for any $a\in R^p$, $-\varepsilon<_K a<_K \varepsilon$. For $a,b\in R^p$, $a\le_K b$ if and only if $a_i=b_i$ for i=1,2,...,p, or $a_i<b_i$ for i=1,2,...,p, and $a<_K b$ if and only if $a_i<b_i$ for i=1,2,...,p. Then we have $a<_K b$ if and only if a=b or a<b. And $-\varepsilon<a<\varepsilon$ for all $a\in R^p$. For any $a\in\underline{R}^p$, $B\subseteq R^p$ and $C\subseteq R^p$, we define: $a+\varepsilon=+\varepsilon$; $a+B=\{a+b\in R^p : b\in B\}$; $B+C=\{b+c\in\underline{R}^p : b\in B, c\in C\}$.The p-dimensional point consisting of a common element $a\in R$ is simply denoted by **a** in this paper. Suppose $X\subseteq R^p$ and $F:R^n\to P(R^p)$, sometimes, we denote F(X) or $\cup_x F(x)$ to be $\cup_{x\in X}F(x)$. Additionally, considering the length of the paper, we omit the proofs of all the theorems.

2. WEAK SUPREMUM AND INFIMUM SETS AND THEIR PROPERTIES

Definition 2.1 Let $Y\subseteq\underline{R}^p$. $y\in\underline{R}^p$ is said to be a weak maximal (minimal) point of Y if $y\in Y$ and there is no $y'\in Y$ satisfying $y<y'$ ($y'<y$). The set of weak maximal (minimal) points of Y is denoted by w-MaxY (w-MinY), which will be called the weak maximal (minimal) set of Y.

Generally, w-MaxY or w-MinY may be empty. It is well known that if Y is a nonempty compact set in R^p, then the w-MaxY and w-MinY are not empty [10].

Definition 2.2 Let $Y\subseteq\underline{R}^p$, S(Y-K) and I(Y+K) are defined by

$$S(Y-K)=\begin{cases}\underline{R}^p \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots R^p\subseteq Y-K, or \varepsilon\in Y\\ -\varepsilon\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots Y=\emptyset or Y=\{-\varepsilon\}\\ cl((Y-K)\cap R^p \ldots\ldots\ldots\ldots otherwise\end{cases}$$

$$I(Y+K)=\begin{cases}\underline{R}^p \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots R^p\subseteq Y+K or -\varepsilon\in Y\\ -\varepsilon\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots Y=\emptyset or Y=\{\varepsilon\}\\ cl((Y+K)\cap R^p \ldots\ldots\ldots\ldots otherwise\end{cases}$$

where for $B\subseteq R^p$, cl(B) denotes the topological closure of B. Furthermore, we shall define w-SupY and w-InfY by

w-SupY=w-Max S(Y-K) ; w-InfY=w-Min I(Y+K) which will be called the weak supremum set and weak infimum set of Y respectively, and their elements will be called the weak supremum point and weak infimum point of Y, respectively.

From the definition of the topological closure, we know that $R^p \subseteq Y+K$ if and only if $R^p \subseteq cl((Y-K) \cap R^p)$, and $R^p \subseteq Y+K$ if and only if $R^p \subseteq cl((Y+K) \cap R^p)$. When p=1, w-SupY={supY} w-InfY={infY}. For $\forall Y \subseteq \underline{R}^p$, w-SupY=-w-Inf(-Y), w-InfY=-w-Sup(-Y),

and if $Y_1 \subseteq Y_2 \subseteq \underline{R}^p$, then $S(Y_1 - K) \subseteq S(Y_2 - K)$, $I(Y_1 + K) \subseteq I(Y_2 + K)$.

We shall now introduce new notations "$\Uparrow$", "$\Rightarrow$", "$\Leftarrow$" and "$\Leftrightarrow$", which will be used repeatedly in this paper.

Let $Y_1, Y_2 \subseteq R^p$, we write $Y_1 \Uparrow Y_2$ if there are no points $y_1 \in Y_1$ and $y_2 \in Y_2$ satisfying $y_2 < y_1$. We write $Y_1 \Rightarrow Y_2$ if $Y_1 \Uparrow Y_2$ and for any $y_1 \in Y_1$ there exists some $y_2 \in Y_2$ such that $y_1 < y_2$. We write $Y_1 \Leftarrow Y_2$ if $-Y_2 \Rightarrow -Y_1$. We write $Y_1 \Leftrightarrow Y_2$ if both $Y_1 \Rightarrow Y_2$ and $Y_1 \Leftarrow Y_2$ hold.

The following lemma is easily verified.

Lemma 2.1 Let $Y_1, Y_2, Y_3 \subseteq R^p$.

(1) Transitivity. If $Y_1 \Rightarrow Y_2$ and $Y_2 \Rightarrow Y_3$, then $Y_1 \Rightarrow Y_3$. If $Y_1 \Leftarrow Y_2$ and $Y_2 \Leftarrow Y_3$, then and $Y_1 \Leftarrow Y3$.

(2) Uniqueness. If $Y_1 \Rightarrow Y_2$ and $Y_2 \Rightarrow Y_1$, then $Y_1 = Y_2$. If $Y_1 \Leftarrow Y_2$ and $Y_2 \Leftarrow Y_1$, then $Y_1 = Y_2$. In each case, Y =w-MaxY =w-MinY ($Y=Y_1$ or Y_2).

(3) Self-conversity. If Y_1 =w-MinY_1 or Y_1 =w-MaxY_1, then $Y_1 \Rightarrow Y_1$ and $Y_1 \Leftarrow Y_1$

Theorem 2.1 (Existence Theorem) Let $Y \in \underline{R}^p$. Then

(1) w-SupY is the nonempty set in $\underline{R}^p$. Furthermore, for each $y \in S(Y-K)$, there exists a point $y' \in$w-SupY such that $y \leq_K y'$. In particular, for each $y \in S(Y-K) \backslash \{-\varepsilon\}$, there exists a unique number $t \in [0,+\infty]$ such that $y+t \in$w-SupY. Similarly, for each $y \notin S(Y-K)$ and $y \neq -\varepsilon$, there exists some $y' \in$w-SupY such that $y' <_K y$. In particular, if $y \neq \varepsilon$, then there exists a unique number $t \in [-\infty,0]$ such that $y+t \in$w-SupY.

(2) w-InfY is the nonempty set in $\underline{R}^p$. Furthermore, for each $y \in I(Y+K)$ there exists $y' \in$w-InfY such that $y' \leq_K y$. In particular, for each $y \in I(Y+K) \backslash \{\varepsilon\}$, there exists a unique number $t \in [-\infty,0]$ such that $y+t \in$w-InfY. Similarly, for each $y \notin I(Y+K)$ there exists $y' \in$w-InfY such that $y' <_K y$. In particular, if $y \neq -\varepsilon$, then there exists a unique number $t \in [0,+\infty]$ such that $y+t \in$w-InfY.

Theorem 2.2 For any $Y \subset \underline{R}^p$, w-MaxY$\subseteq$w-SupY, and w-MinY$\subseteq$w-InfY.

Theorem 2.3 Let $Y \subset \underline{R}^p$. If w-SupY$\neq \pm\{\varepsilon\}$, then $(\text{w-SupY}+int(K))^c_{-\infty} = S(Y-K)$. Similarly, if w-InfY$\neq \pm\{\varepsilon\}$, then $(\text{w-InfY}-int(K))^c_{+\infty} = I(Y+K)$. Here for any $B \subset \underline{R}^p$, we denote $B^c_{-\infty} = \underline{R}^p \backslash (B \cup \{-\varepsilon\})$, $B^c_{+\infty} = \underline{R}^p \backslash (B \cup \{\varepsilon\})$.

Theorem 2.4 Let $Y \subset \underline{R}^p$. If w-SupY$\neq \pm\{\varepsilon\}$, then w-SupY+K is closed in R^p. Similarly, if w-InfY$\neq \pm\{\varepsilon\}$, then w-InfY-K is closed in R^p.

Theorem 2.5 Let $a \in R^p$, $Y \subset \underline{R}^p$, $Y' \subset \underline{R}^p$. Then

(1) w-Sup(Y+a)=a+w-SupY ; W-Inf(Y+a)=a+w-InfY.

(2) If $Y\neq\emptyset$ and $Y\neq\{\varepsilon\}$, then $a\Uparrow Y$ if and only if $a\in$ w-InfY-K. Similarly, if $Y\neq\emptyset$ and $Y\neq\{-\varepsilon\}$, then $Y\Uparrow a$ if and only if $a\in$ w-SupY+K.

(3) If $a\Rightarrow Y'$ and $Y'\Uparrow Y$, then $a\Uparrow Y$.

Theorem 2.6 Let Y_1, $Y_2\subseteq\underline{R}^p$. If Y_1 -K$\subseteq Y_2$ -K, then w-Sup$Y_1\Leftrightarrow$ w-SupY_2. Similarly, f Y_1 +K$\subseteq Y_2$ +K, then w-Inf$Y_1\Leftrightarrow$ w-InfY_2.

Theorem 2.7 Let $Y\subseteq\underline{R}^p$. Then

(1) w-Inf(w-SupY)=w-SupY; w-Sup(w-InfY)=w-InfY.

(2) w-MaxY$\subseteq$w-Inf(w-MaxY); w-MaxY$\subseteq$w-Sup(w-MaxY). Substituting w-Max for w-Min, the conclusions still hold.

Theorem 2.8 Let Y, Y_1, $Y_2\subseteq\underline{R}^p$.

(1) If $Y\Leftrightarrow Y$, then w-SupY $\Leftrightarrow$ w-InfY.

(2) If the following corresponding operations make sense, i.e. there appears no $+\varepsilon-\varepsilon$ form operation, then w-SupY_1 +w-SupY_2, w-Sup(Y_1 +Y_2), hence

w-SupY_1 +w-Sup$Y_2\subseteq$w-Sup(Y_1 +Y_2)+K and w-Sup(Y_1 +Y_2)$\subseteq$w-SupY_1 +w-SupY_2 +K.

Theorem 2.9 Let $Y\subseteq\underline{R}^p$, and $Y'=\{y\in\underline{R}^p : y\Uparrow Y\}$, then w-SupY'=w-InfY. Similarly, let $Y''=\{y\in\underline{R}^p : Y\Uparrow y\}$, then w-InfY''=w-SupY.

Theorem 2.10 Let X be some vector space, $F:X\to P(\underline{R}^p)$ be the point-to-set map from X into $\underline{R}^p$, and denote $F(X)=\cup_{x\in X} F(x)$, then we have

w-SupF(X)=w-Sup($\cup_{x\in X}$ w-SupF(x)) and w-InfF(X)=w-Inf($\cup_{x\in X}$ w-InfF(x)).

Hence, for $Y\subseteq\underline{R}^p$, we have w-Sup(w-SupY)=w-SupY, and w-Inf(w-InfY)=w-InfY

3. W-(H,Ω) CONJUGATE MAPS, W-(H,Ω)-SUBGRADIENTS AND W-(H,Ω)-REGULARITIONS

Throughout this paper, we shall designate X and Y two locally convex Hausdorff spaces (finite-dimensional). Usually, we can suppose that $X=R^n$ and $Y=R^m$. For A being any set of some vector space, and we denote by P(A) the power set of A.

Definition 3.1 Let $F:X\to P(\underline{R}^p)$, Ω be a family of vector-valued functions $\omega:X\to Y$, and H be a family of vector-valued functions $h:Y\to\underline{R}^p$, closed w-Sup pointwise, i.e., if $H'\subseteq H$, then w-Sup{H'}$\subseteq$H, where w-Sup is taken pointwisely, that is, for $\forall y\in Y$, (w-Sup{H'})(y)=w-Sup{H'(y)}. Under the above assumption, the w-(H,Ω) conjugate map w-$F^c:\Omega\to P(H)$ of F is defined by the formula:

w-$F^c(\omega)$=w-Sup$\{h:h\in H, h\otimes\omega\Uparrow F\}$ for $\forall\omega\in\Omega$

that is, $w\text{-}F^c(\omega)(y)=w\text{-}Sup\{h(y): h\in H, h\otimes\omega\Uparrow F\}$ for $\forall\omega\in\Omega$ and $y\in Y$

Inversely, let $G:\Omega\to P(H)$, the w-(H,X) conjugate map $w\text{-}G^*:X\to P(\underline{R}^p)$ of G is defined by the formula:

$$w\text{-}G^*(x)=w\text{-}Sup\{h(\omega(x)): h\in H, \omega\in\Omega, h\Uparrow G(\omega)\} \quad \text{for } \forall x\in X$$

It is easily verified that if $w\text{-}SupG(\omega)=G(\omega)$ for $\forall\omega\in\Omega$, then

$$w\text{-}G^*(x)=w\text{-}Sup \cup_{\omega} G(\omega)(\omega(x)) \quad \text{for } \forall x\in X$$

Moreover, the w-(H,X) conjugate map of $w\text{-}F^c$ is called the w-(H,Ω) biconjugate map of F, and is denoted by $w\text{-}F^{c*}$, i.e., $w\text{-}F^{c*}:X\to P(\underline{R}^n)$, and $w\text{-}F^{c*}=w\text{-}(w\text{-}F^c)^*$, and $w\text{-}F^{c*}(x)=w\text{-}Sup \cup_{\omega\in\Omega} w\text{-}F(\omega)(\omega(x))$ for $\forall x\in X$.

Definition 3.2 (w-ΓpH (Ω) Family) The set of the point-to-set maps $F:X\to P(\underline{R}^n)$ which can be written in the following form:

$$F(x)=w\text{-}Sup\{h(\omega(x)): h\in H',\omega\in\Omega'\} \quad \text{for } \forall x\in X$$

is denoted by $w\text{-}\Gamma^p{}_H(\Omega)$, where $H'\subset H$, $\Omega'\subseteq\Omega$ and Ω' is nonempty.

Definition 3.3 (w-ΓpH (Ω)-normalization) Let $F:X\to P(\underline{R}^n)$. The point-to-set map $\tilde{F}(H,\Omega):X\to P(\underline{R}^n)$ defined by the following formula is called the $w\text{-}\Gamma^p{}_H(\Omega)$-normalization of F:

$$\tilde{F}(H,\Omega)(x)=w\text{-}Sup\{h(\omega(x)): h\in H, \omega\in\Omega, h\otimes\omega\Uparrow F\} \quad \text{for } \forall x\in X$$

Let $H_L=\{h_b:h_b(t)=t+b\ (\forall t\in R^p), b\in\underline{R}^p\}$, $\Omega=\{\omega_*:\omega_*(x)=\langle\langle x,x^*\rangle\rangle\ (\forall x\in X), x^*\in X^*\}$ where X^* is the dual space of X, and $\langle\bullet,\bullet\rangle$ is a bilinear pairing between X and X^*, and $\langle\langle\bullet,\bullet\rangle\rangle=(\langle\bullet,\bullet\rangle,\langle\bullet,\bullet\rangle,\ldots,\langle\bullet,\bullet\rangle)$.

Theorem 3.1 Let $H=H_L$, $Y=R^p$, $F:X\to P(\underline{R}^p)$, and $G:\Omega\to P(H)$. Then $w\text{-}F^c$ can be written : $w\text{-}F^c(\omega)(t)=t\text{-}w\text{-}Sup\{\cup_x\{\omega(x)\text{-}F(x)\}\}$ for $\forall\omega\in\Omega$ and $t\in Y$. Similarly, $w\text{-}G^*$ can be written: $w\text{-}G^*(x)=w\text{-}Sup \cup_{\omega}\{\omega(x)+\underline{A}(\omega)\}$ for $\forall x\in X$, where $\underline{A}(\omega)=w\text{-}InfA(\omega)$, and $A(\omega)$ satisfies $G(\omega)(t)=t+A(\omega)$ for $\forall t\in Y$ and $\omega\in\Omega$. In particular, $w\text{-}F^{c*}$ can be written: $w\text{-}F^{c*}(x)=w\text{-}Sup \cup_{\omega}\{\omega(x)\text{-}\bar{F}(\omega)\}$ for $\forall x\in X$, where $\bar{F}(\omega)=w\text{-}Sup \cup_{\omega}\{\omega(x)\text{-}F(x)\}$ for $\forall\omega\in\Omega$.

We have seen that if $H=H_L$ and $\Omega=\Omega_L$, $w\text{-}F^c$ is none more than the conjugate map discussed in [8] by substituting Max for w-Sup.

The relationships between the w-(H,Ω) conjugate map and $w\text{-}\Gamma^P{}_H(\Omega)$-normalization of F are given as follows.

Theorem 3.2 Let $F:X\to P(\underline{R}^p)$, and $G:\Omega\to P(H)$. Then

(1) $\tilde{F}(H,\Omega)\in w\text{-}\Gamma^P{}_H(\Omega)$, $\tilde{F}(H,\Omega)\Leftarrow F$, $\tilde{F}(H,\Omega)=w\text{-}F^{c*}$, and $w\text{-}G^*\in w\text{-}\Gamma^p{}_H(\Omega)$.

(2) if $F'\in w\text{-}\Gamma^p{}_H(\Omega)$ and $F'\Uparrow F$, then $F'\Leftrightarrow\tilde{F}(H,\Omega)$

(3) $\tilde{F}$ (H,Ω)=F if and only if F∈w-Γ^p_H (Ω).

(4) w-F^c (ω)(ω(x))⇐F(x) for ∀ω∈Ω and x∈X.

Similar discussions can be made for G:Ω→P(H).

Definition 3.4 (w-ΣpH (Ω) Family) The set of the point-to-set maps G:Ω→P(H) which can be written: G(ω)=w-Sup{h: h∈H_ω } for ∀ω∈Ω, where H_ω ⊆H for each ω∈Ω is denoted by w-Σ^p_H (Ω)

Definition 3.5 (w-ΣpH (Ω)-normalization) Let G:Ω→P(H). The point-to-set map $\tilde{G}$ (H,Ω):Ω→P(H) defined by:

$\tilde{G}$ (H,Ω)(ω)=w-Sup{h: h∈H,h⇑G(ω)}=w-Sup{h: h∈H, h(t)⇑G(ω)(t) ∀t∈Y} for ∀ω∈Ω

is called the w-Σ^p_H (Ω)-normalization of G.

Dually to theorem 3.2, we have:

Theorem 3.3 Let G:Ω→P(H), F:X→P($\underline{R}^p$). Then

(1) $\tilde{G}$ (H,Ω)∈w-Σ^p_H (Ω), $\tilde{G}$ (H,Ω)⇐G, w- $\tilde{G}$ * =w-G*, $\tilde{G}$ ⇑ w-G^{*c} , and w-F^c∈w-Σ^p_H(Ω).

(2) if G'∈w-Σ^p_H (Ω) and G'⇑G, then G'⇔ $\tilde{G}$ (H,Ω).

(3) $\tilde{G}$ (H,Ω)=G if and only if G∈w-Σ^p_H (Ω).

(4) if w-SupG(ω)=G(ω) for some ω∈Ω, then G(ω)(ω(x))⇑w-G*(x) for ∀x∈X.

Definition 3.6 F:X→P($\underline{R}^p$) is said to be w-(H,Ω)-subdifferentiable at (x;y) if y∈F(x), x∈X and there exist h∈H and ω∈Ω such that h(ω(x))=y and h⊗ω⇑F. And such ω∈Ω is called the w-(H,Ω)-subgradient of F at (x;y) The set of all the w-(H,Ω)-subgradients of F at (x;y) is called the w-(H,Ω)-subdifferential of F at (x;y), which is denoted by w-∂_Ω F(x;y). Moreover, we define w-∂_Ω F(x)= $\cup_y$ w-∂_Ω F(x;y). If w-∂_Ω F(x;y)≠∅ for any y∈F(x),then we say that F is w-(H,Ω)-subdifferentiable at x.

Definition 3.7 G:Ω→P(H) is said to be w-(H,X)-subdifferentiable at (ω;h), if h∈G(ω), ω∈Ω and there exists x∈X such that h(ω(x))∈w-G*(x), and h⊗ω⇑w-G* . And such x∈X is called the w-(H,X)-subgradient of G at (ω;h). The set of all the w-(H,X)-subgradients of G at (ω;h) is called the w-(H,X)-subdifferential of G at (ω,h), and will be denoted by w-∂_X G(ω;h). Moreover, let w-∂_X G(ω)= $\cup_h$ w-∂_X G(ω;h). If w-∂_X G(ω;h)≠∅ for each h∈G(ω), then we say that G is the w-(H,X)-subdifferentiable at ω.

Theorem 3.4 Let F:X→P($\underline{R}^p$).

(1) ω∈w-∂_Ω F(x;y) if and only if y∈F(x) and y∈w-F^c (ω)(ω(x)).

(2) w-∂_ΩF(x;y)⊆w-∂_Ω w-F^c*(x;y) for ∀x∈X and y∈$\underline{R}^P$.

(3) If w-F^c*(x)⊆F(x), w-F^c ⊆w-F^{c*c} , then w-∂_Ω F(x;y)=w-∂_Ω w-F^c*(x;y) for ∀y∈w-F^c*(x).

(4) Suppose that Ω contains the element 0_Ω satisfying $0_\Omega(x)=0\in Y$ for $\forall x\in X$, and H has the following property: for each $a\in\underline{R}^p$, there exists $h_a \in H$ such that $h_a(0)=a$. Then $0_\Omega \in w\text{-}\partial_\Omega F(x;y)$ if and only if $y\in F(x)\cap w\text{-Min} \cup_x F(x)$.

Corollary Let $F:X\to P(\underline{R}^p)$.

(1) $w\text{-}\partial_\Omega F(x)\subseteq w\text{-}\partial_\Omega\, w\text{-}F^{c}{}^*(x)$ for $\forall x\in X$.

(2) If $w\text{-}F^{c}{}^*(x)\subseteq F(x)$ and $w\text{-}F^c \subseteq w\text{-}F^{c*c}$, then $w\text{-}\partial_\Omega F(x)=w\text{-}\partial_\Omega\, w\text{-}F^{c}{}^*(x)$.

(3) $\omega\in w\text{-}\partial_\Omega F(x)$ if and only if $F(x)\cap w\text{-}F(\omega)(\omega(x))\neq\varnothing$.

(4) Under the condition of theorem 3.10's (4), $0_\Omega \in w\text{-}\partial F(x)$ if and only if $F(x)\cap w\text{-Min} \cup_x F(x)\neq\varnothing$.

Theorem 3.5 Let $G:\Omega\to P(H)$.

(1) If $x\in w\text{-}\partial_X G(\omega;h)$, then $h(\omega(x))\in G(\omega)(\omega(x))\cap w\text{-}G^{*c}(\omega)(\omega(x))$.

(2) If $w\text{-}\partial_X G(\omega;h)\neq\varnothing$, then there exists $h'\in H$ such that $w\text{-}\partial_X\, w\text{-}G^{*c}(\omega;h')\neq\varnothing$.

(3) If $w\text{-}G^{*c}(\omega)\subseteq G(\omega)$, then $w\text{-}\partial_X G(\omega;h)\supseteq w\text{-}\partial\, w\text{-}G^{*c}(\omega;h)$ for $\forall h\in w\text{-}G^{*c}(\omega)$.

Corollary Let $G:\Omega\to P(H)$.

(1) If $w\text{-}\partial_X G(\omega)\neq\varnothing$, then for some $x\in X$, $G(\omega)(\omega(x))\cap w\text{-}G^{*c}(\omega)(\omega(x))\neq\varnothing$.

(2) $w\text{-}\partial_X G(\omega)\subseteq w\text{-}\partial_X\, w\text{-}G^{*c}(\omega)$.

(3) If $w\text{-}G^{*c}(\omega)\subseteq G(\omega)$, then $w\text{-}\partial_X G(\omega)\subseteq w\text{-}\partial_X\, w\text{-}G^{*c}(\omega)$.

The relationship between the w-(H,Ω)-subgradient and w-(H,X)-subgradient is as follows.

Theorem 3.6 Let $F:X\to P(\underline{R}^p)$, $G:\Omega\to P(H)$.

(1) If $\omega\in w\text{-}\partial_\Omega F(x;y)$, then there exists $h\in w\text{-}F^c(\omega)$ such that $x\in w\text{-}\partial_X w\text{-}F^c(\omega;h)$. Inversely, if $F\in w\text{-}\Gamma^p{}_H(\Omega)$ and $x\in w\text{-}\partial_X\, w\text{-}F^c(\omega;h)$, then there exists $y\in\underline{R}^p$ such that $\omega\in w\text{-}\partial_\Omega F(x;y)$.

(2) If $x\in w\text{-}\partial_X G(\omega;h)$, then there exists $y\in\underline{R}^p$ such that $\omega\in w\text{-}\partial_\Omega\, w\text{-}G^*(x;y)$. Inversely, if $w\text{-}G^{*c}(\omega)\subseteq G(\omega)$, $\omega\in w\text{-}\partial_\Omega w\text{-}G^*(x;y)$, then there exists $h\in H$ such that $x\in w\text{-}\partial_X G(\omega;h)$.

In the following, we will discuss the weak saddle-point problem of point-to-set maps. Let point-to-set map L be $L:X\times Y\to P(\underline{R}^p)$, where X and Y are two finite-dimensional locally convex Hausdorff spaces.

Definition 3.8 The point $(x',y')\in X\times Y$ is called the weak saddle-point of map L if there exists $b\in L(x',y')$ such that $L(x',y)\Uparrow b\Uparrow L(x,y')$ for $\forall x\in X$ And $y\in Y$.

When p=1 and $L:X\times Y\to R$, the above definition coincides with that of the saddle-point, in the sense, of a function.

In the following, for convenience, we shall use the notations given by:

$\text{w-Sup}L_x = \text{w-Sup} \cup_y L(x,y)$; $\text{w-Inf}L_y = \text{w-Inf} \cup_x L(x,y)$;

$\text{w-Infw-Sup}L = \text{w-Inf} \cup_x \text{w-Sup}L_x$; $\text{w-Supw-Inf}L = \text{w-Sup} \cup_y \text{w-Inf}L_y$;

$\text{w-Minw-Sup}L = \text{w-Min} \cup_x \text{w-Sup}L_x$; $\text{w-Maxw-Inf}L = \text{w-Max} \cup_y \text{w-Inf}L_y$

By the definition of the weak saddle-point, for $b\in L(x,y)$, (x,y) is the weak saddle-point of L if and only if $b\in(\text{w-Max} \cup_y L(x,y)) \cap (\text{w-Min} \cup_x L(x,y))$ if and only if $b\in(\text{w-Max} \cup_y \text{w-Min} \cup_x L(x,y)) \cap (\text{w-Min} \cup_x \text{w-Max} \cup_y L(x,y))$.

4. W-(H,Ω) DUALITY IN MULTIOBJECTIVE OPTIMIZATION

Let X,U, and Y be three locally convex Hausdorff spaces, where X is called the decision space and U is called the perturbation space. Let Φ be a map from X×U into $\underline{R}^p$, i.e., $\Phi: X\times U \to P(\underline{R}^p)$. We assume that Φ is not identically empty on X×{0}.

Let Ω_X and Ω_U be two families of functions from X into Y and U into Y, respectively. Especially, we denote $\Omega' = \Omega_X \otimes \Omega_U$ and $\Omega = \Omega_U$, where Ω' is a family of functions from X×U into Y and $\omega' \in \Omega'$ if and only if there exist $\omega\in\Omega_X$ and $\theta\in\Omega_U$ such that $\omega' = \omega\oplus\theta$, that is, $\omega'(x,u) = \omega(x) + \theta(u)$ for $\forall(x,u)\in X\times U$. For $\omega' = \omega\oplus\theta$, we sometimes write it $\omega' = (\omega,\theta)$. In particular, we assume that Ω_X contains the element 0_X satisfying $0_X(x) = 0_Y \in Y$, where 0_Y is the zero element of Y. Finally, Let H be a family of functions from Y into $\underline{R}^p$, closed w-Sup pointwise.

Then we consider the following optimization problems with set-valued objective functions:

(MO) The original problem: Find $x\in X$ such that $\Phi(x,0) \cap \text{w-Min} \cup_x \Phi(x,0) \neq \emptyset$.

(MP) The primal problem: Find $x\in X$ such that $\Phi(x,0) \cap \text{w-Inf} \cup_x \Phi(x,0) \neq \emptyset$.

(MD') The w-(H,Ω) conjugate dual problem of (MO): Find $\theta\in\Omega$ such that

$$\text{w-}\Phi^c(0_X,\theta)(\theta(0)) \cap \text{w-Max} \cup_\theta \text{w-}\Phi^c(0_X,\theta)(\theta(0)) \neq \emptyset.$$

(MD) The w-(H,Ω) conjugate dual problem of (MP): Find $\theta\in\Omega$ such that

$$\text{w-}\Phi^c(0_X,\theta)(\theta(0)) \cap \text{w-Sup} \cup_\theta \text{w-}\Phi^c(0_X,\theta)(\theta(0)) \neq \emptyset .$$

We call solutions of (MO), (MP), (MD'), and (MD) in X or Ω weak efficient solutions, respectively, which will be denoted by w-Eff(MO),w-Eff(MP), w-Eff(MD') and w-Eff(MD), respectively. Moreover, denote

$\text{w-Min(MP)} = \text{w-Min} \cup_x \Phi(x,0)$

$\text{w-Inf(MP)} = \text{w-Inf} \cup_x \Phi(x,0)$

$\text{w-Max(MD)} = \text{w-Max} \cup_\theta \text{w-}\Phi^c(0_X,\theta)(\theta(0))$

$\text{w-Sup(MD)} = \text{w-Sup} \cup_\theta \text{w-}\Phi^c(0_X,\theta)(\theta(0))$

Theorem 4.1 w-Eff(MO)=w-Eff(MP) ; w-Eff(MD')=w-Eff(MD).

By virtue of this theorem, we have only to consider the duality between (MP) and (MD) instead of (MO) and (MD'). The first duality is the following weak duality.

Theorem 4.2 (w-(H,Ω) Weak Duality Theorem)

(1) w-Max(MD)$\Uparrow$w-Min(MP)

w-Max(MD)$\Uparrow\Phi(x,0)$ for $\forall x\in X$

w-$\Phi^c(0_X,\theta)(\theta(0))\Uparrow$w-Min(MP) for $\forall\theta\in\Omega$;

w-Max(MD)$\Uparrow\cup_x \Phi(x,0)$; $\cup_\theta$ w-$\Phi^c(0_X,\theta)(\theta(0))\Uparrow$w-Min(MP) .

(2) If $x\in X$ and $\theta\in\Omega$ satisfy w-$\Phi^c(0_X,\theta)(\theta(0))\cap\Phi(x,0)\neq\varnothing$, then $x\in$w-Eff(MP) and $\theta\in$w-Eff(MD).

(3) If $x\in X$ satisfies $\Phi(x,0)\cap$w-Max(MD)$\neq\varnothing$, then $x\in$w-Eff(MP) and w-Min(MP)$\cap$ w-Max(MD)$\neq\varnothing$. Substituting w-Max for w-Sup, the conclusion still holds.

(4) If $\theta\in\Omega$ satisfies w-$\Phi^c(0_X,\theta)(\theta(0))\cap$w-Min(MP)$\neq\varnothing$, then $\theta\in$w-Eff(MD) and w-Min(MP)$\cap$w-Max(MD)$\neq\varnothing$. Substituting w-Min for w-Inf, the conclusion still holds.

Theorem 4.3 w-Sup(MD)$\Leftrightarrow$ w-Inf(MP).

The perturbed problem relative to (MP) and (MD) is given as follows.

(MP_u) The primal perturbed problem: find $x\in X$ such that

$\Phi(x,u)\cap$w-Inf $\cup_x \Phi(x,u)\neq\varnothing$.

(MD_ω) The dual perturbed problem: find $\theta\in\Omega$ such that

w-$\Phi^c(\omega,\theta)(\omega(0)+\theta(0))\cap$w-Sup $\cup_\theta$ w-$\Phi^c(\omega,\theta)(\omega(0)+\theta(0))\neq\varnothing$.

Definition 4.1 The point-to-set maps $P:U\to P(\underline{R}^p)$ and $D:\Omega_X\to P(H)$ defined by

$P(u)$=w-Inf $\cup_x \Phi(x,u)$ for $\forall u\in U$.

$D(\omega)(t)$=w-Sup $\cup_\theta$ w-$\Phi^c(\omega,\theta)(t+\theta(0))$ for $\forall\omega\in\Omega$ and $t\in Y$

are called the primal and dual perturbed maps, respectively.

The following theorem shows how to describe (MP) and (MD) by means of P and D.

Theorem 4.4

(1)w-$P^c(\theta)$=w-$\Phi^c(0_X,\theta)$ for $\forall\theta\in\Omega$, that is, w-$P^c(\theta)(t)$=w-$\Phi^c(0_X,\theta)(t)$ for $\forall\theta\in\Omega$ and $y\in Y$.

(2) w-$D^*(x)$=w-$\Phi^{c*}(x,0)$ for $\forall x\in X$.

(3) If $\Phi\in$w-$\Gamma^p_H(\Omega^*)$, in particular, as long as $\Phi(x,0)$=w-$\Phi^{c*}(x,0)$ for $\forall x\in X$, then w-$D^*(x)=\Phi(x,0)$ for $\forall x\in X$.

ASSUMPTION (A1) For each $a\in\underline{R}^p$, there is $h_a\in H$ such that $h_a(0)=a$.

ASSUMPTION (A2) $\Phi\in$w-$\Gamma^p_H(\Omega^*)$ or Φ is w-$\Gamma^p_H(\Omega^*)$-normalizable at $(x,0)\in X\times U$ for each $x\in X$, i.e., w-$\Phi^{c*}(x,0)=\Phi(x,0)$ for $\forall x\in X$.

Theorem 4.5

(1) w-Sup(MD)=w-P^{c}*(0), w-Inf(MP)=P(0).

(2) Under the assumptions (A1) and (A2), w-Inf(MP)=w-$D^{*c}(0_X)(0)$.

We now introduce two concepts: normality and stability.

Definition 4.2 (MP) is called w-(H,Ω)-stable if P is w-(H,Ω)-subdifferentiable at $u=0\in U$. Dually, (MD) is called w-(H,X)-stable if D is w-(H,X)-subdifferentiable at $\omega=0_X\in\Omega_X$.

Definition 4.3 (MP) is called w-(H,Ω)-normal if P satisfies $P(0)\subseteq$w-P^{c}*(0). Dually, (MD) is called w-(H,X)-normal if D satisfies $D(0_X)(0)\subseteq$w-$D^{*c}(0_X)(0)$.

Theorem 4.6

(1) (MP) is w-(H,Ω)-normal if and only if w-Inf(MP)=w-Sup(MD).

(2) Under assumptions (A1) and (A2), (MD) is w-(H,X)-normal if and only if w-Inf(MP) =w-Sup(MD) if and only if (MP) is w-(H,Ω)-normal.

Theorem 4.7

(1) If w-P^{c}*$(0)\subseteq P(0)$, then w-Eff(MD)=w-∂_Ω P(0).

(2) Under assumptions (A1) and (A2), w-Eff(MP)=w-∂_X w-$D^{*c}(0_X)$. Moreover, if w-D^{*c}*$(0_X)(0)\subseteq D(0_X)(0)$, then w-Eff(MP)=w-$\partial_X$ D(0).

Theorem 4.8

(1) (MP) is w-(H,Ω)-stable if and only if w-Inf(MP)=w-Sup(MD)=w-Max(MD).

(2) Under assumptions (A1) and (A2), (MD) is w-(H,X)-stable if and only if

w-Inf(MP)=w-Sup(MD)=w-Min(MP)

Definition 4.4 The set w-DGS(MP,MD)=w-Inf(MP)∩w-Sup(MD) is called the w-(H,Ω) conjugate dual gap set of (MP) and (MD), briefly called the dual gap set.

From the results of the above several theorems, we have the following w-(H,Ω) strong duality theorem based on the w-(H,Ω)-stability or w-(H,Ω)-subgradient.

Theorem 4.9 (w-(H,Ω) Strong Duality Theorem)

(1) If (MP) is w-(H,Ω)-stable, then (MD) has the solution $\theta\in\Omega$, $\theta\in$w-Eff(MD). Moreover, (MP) is w-(H,Ω)-stable if and only if for each $y\in$w-Inf(MP), there is $\theta\in$w-Eff(MD) such that $y\in$w-$\Phi^c(0_X,\theta)(\theta(0))$. And if (MP) is w-(H,Ω)-stable and w-Min(MP)≠∅, then for each $x\in$w-Min(MP), there is $\theta\in\Omega$ such that

$$\Phi(x,0)\cap \text{w-}\Phi^c(0_X,\theta)(\theta(0))\neq\emptyset .$$

(2) (MD) has a solution, i.e., w-Eff(MD)≠∅, and w-DGS(MP,MD)=w-Inf(MP) or

w-DGS(MP,MD)=w-Sup(MD), if and only if (MP) is w-(H,Ω)-stable. In this case,

w-Eff(MD) =w-∂_Ω P(0).

Theorem 4.10 (w-(H,Ω) Inverse Duality Theorem)

Under assumptions (A1) and (A2), we have

(1) (MD) is w-(H,X)-stable if and only if for each y∈w-Sup(MD), there exists x∈w-Eff(MP) such that y∈Φ(x,0). In particular, if (MD) is w-(H,X)-stable, then w-Eff(MP)≠∅, and for each θ∈w-Eff(MD), there is x∈w-Eff(MP) such that w-$\Phi^c(0_X,\theta)$ (θ(0))∩Φ(x,0)≠∅ .

(2) w-Eff(MP)≠∅ and w-DGS(MP,MD)=w-Inf(MP) or w-Sup(MD) if and only if (MD) is w-(H,X)-stable. In this case, w-Eff(MP)=w-∂_X D(0_X).

Corollary Under assumptions (A1) and (A2), (MP) is w-(H,Ω)-stable and w-Eff(MP) ≠∅ if and only if (MD) is w-(H,X)-stable and w-Eff(MD)≠∅. In this case, w-Inf(MP) =w-Sup(MD)=w-Max(MD)=w-Min(MP).

5. W-(H,Ω)-LAGRANGIAN MAPS

In this section, we shall introduce a w-(H,Ω)-Lagrangian map of the (MP) relative to the given perturbation Φ, and clarify the relationship between pairs of (MP) and (MD) and the weak saddle-points of the w-(H,Ω)-Lagrangian map.

Definition 5.1 The w-(H,Ω)-Lagrangian map L:X×Ω→P($\underline{R}^p$) of (MP) relative to Φ is defined by L(x,θ)=w-Φ_x^c (θ)(θ(0)) for ∀x∈X and θ∈Ω, where for each x∈X, Φ_x : Ω→P($\underline{R}^p$), Φ_x(u)=Φ(x,u) for ∀u∈U.

The (MP) and (MD) are represented by the w-(H,Ω)-Lagrangian map L as follows.

Theorem 5.1

(1) w-$\Phi^c(0_X,\theta)$(θ(0))⇔w-InfL_θ for ∀θ∈Ω, If H is closed w-Inf pointwise, then for ∀θ∈Ω, w-$\Phi^c(0_X,\theta)$(θ(0))=w-Inf $\cup_x$ L(x,0) =w-InfL_θ .

(2) w-SupL_x ⇔ w-InfΦ(x,0) for ∀x∈X. If Φ_x ∈w-$\Gamma^p{}_H$ (Ω), then w-SupL_x =Φ(x,0). If Φ_x ∈w-$\Gamma^p{}_H$ (Ω) for ∀x∈X, then for ∀x∈X, w-SupL_x =Φ(x,0).

Theorem 5.2 If H is closed w-Inf pointwise, and Φ_x ∈w-$\Gamma^p{}_H$ (Ω) for ∀x∈X, then the following conditions are equivalent to each other:

(1) (x,θ)∈X×Ω is a weak saddle-point of L;

(2) x∈w-Eff(MP), θ∈w-Eff(MD) and Φ(x,0)∩w-$\Phi^c(0_X,\theta)$(θ(0))≠∅;

(3)Φ(x,0)∩w-$\Phi^c(0_X,\theta)$(θ(0))≠∅.

Theorem 5.3 Under the Condition of theorem 5.2, if (MP) is w-(H,Ω)-stable, then the following conditions are equivalent to each other:

(1) $x \in$ w-Eff(MP);

(2) there exists $\theta \in \Omega$ such that (x,θ) is a weak saddle-point of L. in this case, $\theta \in$ w-Eff (MD).

Theorem 5.5 Under the condition of theorem 5.2, if assumptions (A1) and (A2) hold, and (MD) is w-(H,X)-stable, then the following conditions are equivalent to each other:

(1) $\theta \in$ w-Eff(MD);

(2) there exists $x \in X$ such that (x,θ) is a weak saddle-point of L. in this case, $x \in$ w-Eff (MP).

6. APPLICATIONS FOR SEVERAL SPECIAL CASES OF H AND Ω

CASE 1. Vector Dual variable.

Take $H=H_L$, $\Omega_X=\{\langle\langle x^*,\bullet\rangle\rangle: x^* \in X^*\}$, $\Omega_U=\{\langle\langle u^*,\bullet\rangle\rangle: u^* \in U^*\}$

where X^*, U^* are the dual space of X and U, respectively, X and X^* or U and U^* are placed in duality by a linear pairing denoted by $\langle\bullet,\bullet\rangle$.

In this special case for H and Ω, the results of [6 ,7] are none more than that of this paper.

CASE 2. Matrix Dual Variable.

Consider the following original problem: w-Min$\{f(x): x \in X_0\}$

where $X_0=\{x \in X: G(x) \leq_Q 0\}$ and $X \subseteq R^n$, and

(1) Q is the pointed closed convex cone in R^m with the nonempty interior $\text{int}(Q) \neq \emptyset$;

(2) $f: R^n \to R^p$, continuous and convex;

(3) $g: R^n \to R^m$, continuous and Q-convex.

Let

$$\Phi(x,u)=\begin{cases} f(x)+\partial K \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots x \in X, g(x) \leq_Q u, u \in R^m \\ +\infty \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots otherwise \end{cases}$$

and $H=H_L$, $\Omega=\{\Lambda \in R^{p\times m} : \Lambda Q \subseteq K\}$, where ∂K denotes the boundary of K.

In this special case, the results of this paper reduce to these of chapter 6 in [8].

7. CONCLUSIONS

In this paper, based on weak efficiency, we have extended the concepts of "conjugate", and "subdifferentiability" of functions to those of point-to-set maps, and developed a duality theory in multiobjective optimization.

One possible generalization of this paper is to consider the other kind of "K", for example, letting K=int(D)∪{0}, where D is a convex cone with nonempty interior. This form of K will be useful to develop the duality theory for multiobjective optimization with the domination structure. [4] has discussed this generalization.

Finally, we hope to make contributions for solving multiobjective optimization problems in some sense by taking special cases for H and Ω.

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