Then, let $\varepsilon > 0$ be arbitrary, it follows that
\[
|\rho|_{\varepsilon} \leq \varepsilon + M \sum_{k=1}^{n} |\rho|_{\varepsilon}
\]
which together with (4) and the arbitrariness of $\varepsilon$ imply that
\[
|\rho|_{\varepsilon} \to 0.
\]
Thus, $h$ is continuous.

Accordingly, applying Lemma 2.4 we conclude that $F$
\[
\begin{align*}
\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) &\in F(t,x,u(t,x),u_x(x)), \quad (t,x) \in [0,1] \times [0,\pi], t \neq t_i, \\
u(t_i,x) - u(t_i,x) &= \frac{1}{pm} \sin(\pi x), \quad t_i = \frac{k}{m+1}, k = 1, \ldots, m, \\
u(t,0) &= \sin(\pi x), \quad t \in [0,1], \\
u(t,x) &= \varphi(x,t), \quad t \in [-h,0], x \in [0,\pi],
\end{align*}
\]
where $p>1$. $F(t,x,u,v) = [f_1(t,x,u,v), f_2(t,x,u,v)]$ for each $(t,x,u,v) \in [0,1] \times [0,\pi] \times PC_0$.

Let $A : D(A) \subset X \to X$ be operator defined by $A\omega = \frac{\partial^2 \omega}{\partial x^2}$ with domain $D(A) = \{\omega \in X; \omega'\omega\}$ are absolutely continuous, $\omega'' \in X$ and $\omega(0) = \omega(\pi) = 0$.

It is known that $A$ has a discrete spectrum and the eigenvalues are $-n^2, n \in N$, with the corresponding normalized eigenvectors $\sigma_n(x) = n \sin(nx), 0 \leq x \leq \pi$.

Moreover, $A$ generates a compact, analytic semigroup $\{T(t)\}_{t \geq 0}$ on $X$:
\[
T(t)\omega = \sum_{n=1}^{\infty} e^{-nt} (\omega, \sigma_n) \sigma_n, \quad \|T(t)\|_{L(X)} \leq e^{-t} \leq 1 \quad \text{for all } t \geq 0
\]
(see Henry, 1981). According to the compactness of $T(t)$ for $t > 0$, one can verify that $T(t)$ is uniform operator topology continuous for $t > 0$.

To treat the system (10), we assume that the functions $f_1, f_2 \in [0,1] \times [0,\pi] \times PC_0$ satisfy
\[
(F_1) f_1(t,x,u,v) \in \mathbb{R} \times \mathbb{R}^{+},
\]
and each $(t,x,u,v) \in [0,1] \times [0,\pi] \times PC_0$.
\[
(F_2) f_2 \text{ is } L.s.c \text{ and } f_1 \text{ is } u.s.c.;
\]
\[
(F_3) \text{ there exist functions } \eta_1, \eta_2 \in L^{\infty}(\mathbb{R}, \mathbb{R}^+) \text{ such that } \|f(t,x,u,v)\| \leq \eta_1(t) \|u\| + \eta_2(t);
\]
Then one can verify (see Chen, 2013; Vrabie, 2012) that the multi-valued function $F : [0,1] \times [0,\pi] \to \mathbb{R}^2$ defined as
\[
F(t,u,v) = \{x \in X \mid x \in [f_1(t,x,u(x,v), f_2(t,x,u(x,v)] \}
\]
satisfies assumptions $(H_1) - (H_2)$ (with $\eta(t) = \sqrt{\pi} \max \{\eta_1(t), \eta_2(t)\}$ in $(H_2)$).

Define
\[
I_k(u(t_k))(x) = \frac{1}{pm} \sin(\pi x),
\]
has at least one fixed point in $\mathcal{D}$, which is a mild solution of (1). The proof is complete.

3. AN EXAMPLE OF EXISTENCE RESULT

Take $X = L^1(0,\pi)$ and denote its norm by $\|\|$, and inner product by $(\cdot, \cdot)$ to illustrate our abstract results, let us consider the system of partial differential inclusion in the form
\[
\begin{align*}
\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) &\in F(t,x,u(t,x),u_x(x)), \quad (t,x) \in [0,1] \times [0,\pi], t \neq t_i, \\
u(t_i,x) - u(t_i,x) &= \frac{1}{pm} \sin(\pi x), \quad t_i = \frac{k}{m+1}, k = 1, \ldots, m, \\
u(t,0) &= \sin(\pi x), \quad t \in [0,1], \\
u(t,x) &= \varphi(x,t), \quad t \in [-h,0], x \in [0,\pi],
\end{align*}
\]
It is clear that
\[
\|I_k(y)\| \leq \frac{\sqrt{\pi}}{pm}, \quad k = 1, \ldots, m, y \in X, \\
\|I_k(y_1) - I_k(y_2)\| \leq \frac{1}{pm} \|y_1 - y_2\|, \quad k = 1, m, y_1, y_2 \in X.
\]
These yield that the hypotheses $(H_2)$ are satisfied.

Assume that $F$ satisfies $(H_3)$ with $\int_0^1 \mu(s) ds < \frac{p-1}{4\mu}$, then all the conditions in Theorem 3.1 are satisfied. Hence, the system (10) has at least one mild solution.

REFERENCES


