converge uniformly to \( u \) in \( PC_T \). Subsequently, the lemma can be proved in an argument similar to Lemma 3.1 of Chen (2013).

**Remark 3.2.** From \((H_5)\), it follows that there exist numbers \( l_i \geq 0, k = 1, \ldots, m \), such that

\[
\chi(I(D)) \leq l_i \chi(D)
\]

for any bounded set \( D \subset X \).

**Theorem 3.1.** Let the assumptions \((H_1)-(H_6)\) hold.

\[
D_0 = \{ u \in PC_T : u(t) = \varphi(t) \text{ for } t \in [-h,0], \text{ and } \sup_{a \in [0,r]} \| u(s) \| \leq \omega(t) \text{ for all } t \in J \},
\]

where \( \omega \) is the solution of the following integral equation

\[
\omega(t) = N_1 + 2M \int_0^t \eta(s) \omega(s) ds,
\]

where

\[
N_1 = M \| \varphi(0) \| + M \sum_{k=1}^{m} c_k (M + M \| \varphi \|_q) \| \eta \|_{\lambda_j(X^*)}.
\]

One can find that \( D_0 \subset PC_T \) is closed, bounded and convex.

Let us define the multi-valued map \( \mathcal{F} : PC_T \rightarrow 2^{PC_T} \) as follows:

\[
\mathcal{F} = \mathcal{G} \circ Sel_u,
\]

where \( \mathcal{G}(f) \) is the unique mild solution of the problem (3) corresponding to \( f \in L(J; X) \). In fact, \((H_5)\) and \((4)\) ensure the uniqueness of the mild solution of the problem (3).

We first claim that \( \mathcal{F}(D_0) \subset D_0 \). Indeed, taking \( u \subset D_0 \) and \( v \in \mathcal{F}(D_0) \), there exists \( f \in Sel_u(u) \) such that \( v = \mathcal{G}(f) \).

For each \( t \in J \), it follows from \((H_2)\) and \((H_6)\) that

\[
\| v(t) \| \leq M \| \varphi(0) \| + M \| f(t) \| + M \sum_{s \in \Delta(s)} I_h(u(t_s)) \leq N_1 + 2M \int_0^t \eta(s) \sup_{s \in [0,t]} \| u(s) \| ds + \frac{M}{\lambda_j(Y)} \sum_{s \in \Delta(s)} I_h(u(t_s)) = \omega(t),
\]

where \( |u|_{\lambda_j} = \sup_{s \in [0,t]} \| u(s) \| \leq \| u \|_{[]} + \sup_{s \in [0,t]} \| u(s) \| \).

This implies \( v \in D_0 \), and then one has \( \mathcal{F}(D_0) \subset D_0 \).

Let \( \mathcal{D}_{n+1} = \text{conv} \mathcal{F}(D_0) \), \( n' = 0, 1, 2, \ldots \),

\[
\chi(T(t-s)f_n(s)) \leq M \mu(s) \chi(u_n(s)) + \sup_{s \in [-h,0]} \chi(u_n(s+\sigma)) \leq 2M \mu(s) \chi(u_n(s)).
\]

Then, by (2), (7), Lemma 2.3 and Remark 3.2, one obtains that for each \( t \in J \),

\[
\chi(u_n(t)) \leq \chi\left( \int_0^t \chi(T(t-s)f_n(s)) ds \right) + \chi\left( \sum_{s \in \Delta(s)} T(t-t_s)I_h(u_n(t_s)) \right) \leq 4M \| \mu \|_{\lambda_j(X^*)} + M \sum_{s \in \Delta(s)} I_h(u_n(s)) = 4M \| \mu \|_{\lambda_j(X^*)} + M \sum_{s \in \Delta(s)} I_h(u_n(s)).
\]

Assume further that \( X \) is reflexive and \( T(t) \) is uniform operator topology continuous for \( t > 0 \), then for each \( \varphi \in C_h \), problem (1) has at least one mild solution provided that

\[
4M \| \mu \|_{\lambda_j(X^*)} + M \sum_{k=1}^{m} l_i < 1.
\]

Proof. For \( \varphi \in C_h \), let

\[
D_0 = \{ u \in PC_T : u(t) = \varphi(t) \text{ for } t \in [-h,0], \text{ and } \| u(s) \| \leq \omega(t) \text{ for all } t \in J \},
\]

where \( \omega \) is the solution of the following integral equation

\[
\omega(t) = N_1 + 2M \int_0^t \eta(s) \omega(s) ds,
\]

where

\[
N_1 = M \| \varphi(0) \| + M \sum_{k=1}^{m} c_k (M + M \| \varphi \|_q) \| \eta \|_{\lambda_j(X^*)}.
\]

One can find that \( D_0 \subset PC_T \) is closed, bounded and convex.

Let us define the multi-valued map \( \mathcal{F} : PC_T \rightarrow 2^{PC_T} \) as follows:

\[
\mathcal{F} = \mathcal{G} \circ Sel_u,
\]

where \( \mathcal{G}(f) \) is the unique mild solution of the problem (3) corresponding to \( f \in L(J; X) \). In fact, \((H_5)\) and \((4)\) ensure the uniqueness of the mild solution of the problem (3).

We first claim that \( \mathcal{F}(D_0) \subset D_0 \). Indeed, taking \( u \subset D_0 \) and \( v \in \mathcal{F}(D_0) \), there exists \( f \in Sel_u(u) \) such that \( v = \mathcal{G}(f) \).

For each \( t \in J \), it follows from \((H_2)\) and \((H_6)\) that

\[
\| v(t) \| \leq M \| \varphi(0) \| + M \| f(t) \| + M \sum_{s \in \Delta(s)} I_h(u(t_s)) \leq N_1 + 2M \int_0^t \eta(s) \sup_{s \in [0,t]} \| u(s) \| ds + \frac{M}{\lambda_j(Y)} \sum_{s \in \Delta(s)} I_h(u(t_s)) = \omega(t),
\]

where \( |u|_{\lambda_j} = \sup_{s \in [0,t]} \| u(s) \| \leq \| u \|_{[]} + \sup_{s \in [0,t]} \| u(s) \| \).

This implies \( v \in D_0 \), and then one has \( \mathcal{F}(D_0) \subset D_0 \).

Let \( \mathcal{D}_{n+1} = \text{conv} \mathcal{F}(D_0) \), \( n' = 0, 1, 2, \ldots \),

\[
\chi(T(t-s)f_n(s)) \leq M \mu(s) \chi(u_n(s)) + \sup_{s \in [-h,0]} \chi(u_n(s+\sigma)) \leq 2M \mu(s) \chi(u_n(s)).
\]

Then, by (2), (7), Lemma 2.3 and Remark 3.2, one obtains that for each \( t \in J \),

\[
\chi(u_n(t)) \leq \chi\left( \int_0^t \chi(T(t-s)f_n(s)) ds \right) + \chi\left( \sum_{s \in \Delta(s)} T(t-t_s)I_h(u_n(t_s)) \right) \leq 4M \| \mu \|_{\lambda_j(X^*)} + M \sum_{s \in \Delta(s)} I_h(u_n(s)) = 4M \| \mu \|_{\lambda_j(X^*)} + M \sum_{s \in \Delta(s)} I_h(u_n(s)).
\]