Oscillation for a High Order Nonlinear Difference Equations

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In this paper, we consider certain nonlinear difference equations

\[ \Delta^2(|y_n|^{p-1}\Delta^2 y_n) + q_n|y_{n+1}|^{\beta-1} y_{n+1} = 0, \]

where

(a) \( \alpha, \beta \) are positive constants;
(b) \( \{q_n\}_{n=0}^\infty \) are positive real sequences.

Oscillation and nonoscillation theorems of the above equation is obtained.

**Key words:** Nonlinear difference equations; Oscillation; Nonoscillation; High order

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**Abstract**

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**INTRODUCTION**

In this paper, we consider certain quasilinear difference equations

\[ \Delta^2(|y_n|^{p-1}\Delta^2 y_n) + q_n|y_{n+1}|^{\beta-1} y_{n+1} = 0, \]

where

(a) \( \alpha, \beta \) are positive constants;
(b) \( \{q_n\}_{n=0}^\infty \) are positive real sequences.

The equation which can also be expressed as

\[ \Delta^2(|y_n|^{p-1}\Delta^2 y_n) + q_n|y_{n+1}|^{\beta-1} y_{n+1} = 0, \]

in terms of the asterisk notation

\[ \xi_{\alpha}^* = |\xi|^\alpha \text{sgn } \xi = |\xi|^\alpha \xi, \xi \in R, \alpha > 0. \]

It is clear that if \( \{y_n\} \) is a eventually positive solution of (1), then \( -\{y_n\} \) is a eventually negative solution of (1).

**Lemma 1.1.** Assume that \( \{y_n\} \) is a eventually positive solution of (1), then one of the following two cases holds for all sufficiently large \( n \):

I : \( \Delta y_n > 0, \Delta^2 y_n > 0, \Delta(\Delta^2 y_n)^{\alpha} > 0, \)
II : \( \Delta y_n > 0, \Delta^2 y_n < 0, \Delta(\Delta^2 y_n)^{\alpha} > 0. \)

**Proof.** From (1.1), we have \( \Delta^2(\Delta^2 y_n)^{\alpha} < 0 \) for all large \( n \). It follows that \( \Delta y_n, \Delta^2 y_n, \Delta(\Delta^2 y_n)^{\alpha} \) are eventually monotonic and one-signed.

a) if \( \Delta(\Delta^2 y_n)^{\alpha} < 0 \) eventually. Then combining this with \( \Delta^2(\Delta^2 y_n)^{\alpha} < 0 \), we see that \( \lim_{n \to \infty} (\Delta^2 y_n)^{\alpha} = -\infty \). That is \( \Delta^2 y_n \to \infty \) for all large \( n \). It follows that \( \Delta y_n \to \infty, y_n \to \infty \), which contradicts the positivity of \( \{y_n\} \).

b) if \( \Delta(\Delta^2 y_n)^{\alpha} > 0 \) eventually. Then combining this with \( \Delta^2(\Delta^2 y_n)^{\alpha} < 0 \), we see that \( \Delta^2(\Delta^2 y_n)^{\alpha} \to 0 \) or \( \to a > 0 \) so

\[ (\Delta^2 y_n)^{\alpha} = (\Delta y_n)^{\alpha} + \sum_{i=1}^{\infty} (\Delta^2 y_n)^{\alpha}. \]

If \( (\Delta^2 y_n)^{\alpha} > 0 \), That is \( \Delta^2 y_n > 0 \) is increasing and \( \to C \) or \( \infty \). It follows that \( \Delta y_n > 0 \) if \( (\Delta^2 y_n)^{\alpha} > 0 \), that is \( \Delta^2 y_n < 0 \) is increasing and \( \to d \) or \( 0 \). If \( \Delta y_n < 0 \) Then \( y_n \to \infty \), it is impossible, so \( \Delta y_n > 0 \). This complete the proof of the lemma.

From Lemma (1.1), we know \( \Delta y_n, \Delta^2 y_n, \Delta(\Delta^2 y_n)^{\alpha} \) tend to finite or infinite limits as \( n \to \infty \). Let

\[ \lim_{n \to \infty} \Delta^2 y_n = \omega_i, \text{ i = 0,1,2 } \text{ and } \lim_{n \to \infty} \Delta(\Delta^2 y_n)^{\alpha} = \omega_j. \]

It is that \( \omega_i \) is a finite nonnegative number. One can easily show that:

If \( y_n \) satisfies I, then the set of its asymptotic values \( \omega_i \) falls into one of the following three cases:

I: \( \omega_0 = \omega_1 = \omega_2 = \infty, \omega_3 \in (0, \infty) \),
II: \( \omega_0 = \omega_1 = \omega_2 = \infty, \omega_3 \in (0, \infty) \),
III: \( \omega_0 = \omega_1 = \omega_2 = \omega_3 \in (0, \infty) \).

If \( y_n \) satisfies II, then the set of its asymptotic values \( \omega_i \) falls into one of the following three cases:

II: \( \omega_0 = \omega_1 \in (0, \infty) \), \( \omega_2 = \omega_3 = \omega_4 = \omega_5 = 0 \),
II: \( \lim_{n \to \infty} \frac{y_n}{n} = \text{const} > 0 \),
II': \( \lim_{n \to \infty} \frac{y_n}{n} = 0 \), \( \lim y_n = \infty \),
II'' : \( \lim y_n = \text{const} \).

Let \( y_n \) be a positive solution of (1), such that \( y_n > 0 \), \( y_n(n) > 0 \) for \( n \ge N > n_0 \). Summing (1) from \( n \) to \( \infty \) gives

\[
\Delta(\Delta^2 y_n)(n) = \omega_3 + \sum_{s=0}^{\infty} q_s(y_{\tau(s)})^\beta, \quad n \geq N. \tag{2}
\]

If \( y_n \) is a solution of type \( I_i \), then, first summing (1) from \( n \) to \( \infty \) and then summing the resulting equation three times over \([N, n - 1]\) to obtain

\[
y_n = k_0 + k_1(n - N) + \sum_{s=N}^{n-1}(n-s)[(k_2^a + \sum_{r=s}^{\infty}(\omega_3 + \sum_{\sigma=0}^{r-s} q_\sigma(y_{\tau(s)})^\beta)]^\frac{1}{\beta}, \quad n > N. \tag{3}
\]

A representation for a solution \( y_n \) of type \( II_1 \) is derived by summing (2) with \( \alpha = 0 \). Twice from \( n \) to \( \infty \) and then once from \( N \) to \( n - 1 \) : 

\[
y_n = k_0 + \sum_{s=N}^{n-1}(\omega_1 + \sum_{\tau=1}^{s-N}(r-s)q_\tau(y_{\tau(s)})^\beta] \frac{1}{\beta}, \quad n > N, \tag{4}
\]

A representation for a solution \( y_n \) of type \( II_2 \) is given by (5) with \( \alpha = 0 \). A representation for a solution \( y_n \) of type \( II_3 \) is derived by summing (2) with \( \alpha = 0 \) three times from \( n \) to \( \infty \) yield 

\[
y_n = \omega_0 - \sum_{s=n}^{\infty}(s-n)[\sum_{\tau=1}^{s-N}(r-s)q_\tau(y_{\tau(s)})^\beta] \frac{1}{\beta}, \quad n > N \tag{6}
\]

**MAIN RESULTS**

**Theorem 1.** The Equation (1) has a positive of type \( I_1 \), if and only if

\[
\sum_{s=0}^{n}(s-n)(r-s)^{2/\beta} \beta < \infty. \tag{7}
\]

**Proof.** Necessary. Suppose that (1) has a positive of

\[
G(n, N) = \sum_{s=0}^{n}(s-n)(s-n)^{2/\beta} \beta = \frac{a^2}{(\alpha + 1)(2\alpha + 1)}(n - N)^{1/\beta}\text{,} \quad n \geq N
\]

\[
G(n, N) = 0 \quad \text{for} \quad n < N.
\]

Let \( B_\alpha \) be the Banach space of all real sequences \( Y = \{y_n\} \), with the norm \( \|Y\| = \sup_{n \geq n_0} |y_n| < \infty \), we define a closed, bounded and convex subset \( \Omega \) of \( B_\alpha \) as follows:

\[
\Omega = \{Y = \{y_n\} \in B_\alpha : kG(n, N) \leq y_n \leq 2kG(n, N), \quad n \geq N\}
\]

Define the map \( T: \Omega \to B_\alpha \) as follows:
Oscillation for a High Order Nonlinear Difference Equations

\[ T y_n = \sum_{s=0}^{n-1} (n-s) \left( \sum_{\sigma=r}^{N} \frac{k^\sigma + \sum_{\sigma=r}^{N} q_\sigma (y_{r(\sigma)})^\beta}{2} \right)^{\frac{1}{\alpha}}, \quad n \geq N \]  
\[ T y_n = T y_N \]

\[ \text{a)} \text{ T maps } \Omega \text{ into } \Omega \text{ for } y_n \in \Omega, \text{ then for } n \geq N \]
\[ T y_n \geq K \sum_{s=0}^{n-1} (s-N)^{\frac{1}{\alpha}} = kG(n, N). \]

and

\[ T y_n \leq \sum_{s=0}^{n-1} (n-s) \left( \sum_{\sigma=r}^{N} \frac{k^\sigma + \sum_{\sigma=r}^{N} q_\sigma (2k \tau(\sigma), N^\beta)}{2} \right)^{\frac{1}{\alpha}} \]
\[ \leq \sum_{s=0}^{n-1} (n-s) \left( \sum_{\sigma=r}^{N} k^\sigma + \left( \frac{2k \alpha^2}{(\alpha + 1)(2\alpha + 1)} \right) \sum_{\sigma=r}^{N} q_\sigma (\tau(\sigma)) \right)^{\frac{1}{\alpha}} \]
\[ \leq 2k \sum_{s=0}^{n-1} (n-s)(s-N)^{\frac{1}{\alpha}} = 2kG(n, N). \]

b) \( T \) is continuous. Let \( y^{(k)} \in \Omega \subseteq \Omega \) such that 
\[ \lim_{k \to \infty} \| y^{(k)} - y \| = 0, \]

by using Lebesgue’s dominated convergence theorem, we can conclude that
\[ \lim_{k \to \infty} \left[ T y^{(k)} - T y \right] = 0. \]

c) \( T \) is uniformly-cauchy, \( \forall n_1, n_2 > N \),
\[ \left| T y_{n_1} - T y_{n_2} \right| = \sum_{s=0}^{n_1-1} (n-s) \left( \sum_{\sigma=r}^{N} \frac{k^\sigma + \sum_{\sigma=r}^{N} q_\sigma (y_{r(\sigma)})^\beta}{2} \right)^{\frac{1}{\alpha}} \]
\[ \leq \sum_{s=0}^{n_1-1} (n-s) \left( \sum_{\sigma=r}^{N} k^\sigma + \left( \frac{2k \alpha^2}{(\alpha + 1)(2\alpha + 1)} \right) \sum_{\sigma=r}^{N} q_\sigma (\tau(\sigma)) \right)^{\frac{1}{\alpha}} \]
\[ = \sum_{s=0}^{n_1-1} (n-s) \left( \sum_{\sigma=r}^{N} k^\sigma + \sum_{\sigma=r}^{N} q_\sigma (y_{r(\sigma)})^\beta \right)^{\frac{1}{\alpha}} \]

Therefore, by the Schauder fixed point theorem, there exists a fixed \( T y = y \), which satisfies (1). This completes the proof.

**Theorem 2.** The Equation (1) has a positive of type \( -I_1 \), if and only if
\[ \sum_{a=0}^{n_0} q_a (\tau(n))^2 \beta < \infty. \]

**Proof.** Necessatory. Suppose that (1) has a positive of type \( -I_1 \). Then, it satisfies (4) for \( n \geq N \), which implies
\[ T y_n = \sum_{s=0}^{n-N} (n-s)(s-N)^{\frac{1}{\alpha}} = kG(n, N). \]

The proof is similar to that of theorem 1 and there exists an element \( y = T y \), which is a type \( -I_1 \), solution of (1) with the property that \( \lim_{n \to \infty} \Delta^2 y_n = 2k > 0 \), this completes the proof.

**Theorem 3** The Equation (1) has a positive of type \( -II_1 \), if and only if
\[ \sum_{n-\infty}^{n-N} (n-N)q_n (y_{r(\sigma)})^\beta < \infty. \]

**Proof.** Necessatory. Suppose that (1) has a positive of type \( -II_1 \). Then, it satisfies (4) for \( n \geq N \), which implies
\[ \sum_{a=0}^{n_0} q_a (y_{r(\sigma)})^\beta < \infty. \]
This together with the asymptotic relation
\[ \lim_{n \to \infty} \frac{y_n}{n} = \text{const} > 0 \]
shows that the condition (12) is satisfied.

Sufficiently. Suppose now that (12) holds. Let \( k > 0 \) be any given constant.

Choose \( N > n_0 \) large enough so that
\[ \sum_{n=N}^{\infty} \sum_{r=s}^{n} (s-n)q_r (y_{r(s)})^\beta < 2^{-\frac{\alpha}{\beta}} k \frac{1}{2} \alpha . \]

Put \( N_* = \min \{ N, \inf r(n) \} \), Let \( B_N \) be the Banach space of all real sequences
\[ Y = \{ y_n \} , \]
with the norm \( ||Y|| = \sup_{n \in N} |y_n| < \infty \), we define a closed, bounded and convex subset \( \Omega \) of \( B_N \) as follows:
\[ \Omega = \left\{ Y = \{ y_n \} \in B_N \quad \left| k \leq y_n \leq k \right. \right. \quad \left. \left. n \geq N_* \right\} \right. \]

Define the map \( T : \Omega \to B_N \) as follows:
\[ T_y = k - \frac{1}{2} \sum_{n=N_*}^{\infty} (s-n)q_r (y_{r(s)})^\beta < n \geq N \]

The proof is similar to that of theorem 1 and there exists an element \( y \) such that \( y = T_y \), which is a type-II solution of (1) with the property that \( \lim_{n \to \infty} \Delta y_n = k > 0 \), this completes the proof.

**Theorem 5** The Equation (1) has a positive of type \(-I_2\)
if and only if
\[ \sum_{n=N_*}^{\infty} \left( \frac{1}{n} \right)^{\alpha} < \infty . \]

Proof. Necesssary. Suppose that (1) has a positive of type \(-II_3\). Then, it satisfies (6) for \( n \geq N \), which implies that
\[ \sum_{n=N_*}^{\infty} \left( \frac{1}{n} \right)^{\alpha} < \infty . \]

This together with the asymptotic relation
\[ \lim_{n \to \infty} \frac{y_n}{n} = \text{const} > 0 \]
shows that the condition (14) is satisfied.

Sufficiently. Suppose now that (14) holds. Let \( k > 0 \) be any given constant.

\[ \Omega = \left\{ Y = \{ y_n \} \in B_N \quad \left| \frac{1}{2^{1/\alpha}} (n-N)^{\frac{1}{\alpha}} \leq y_n \leq n^{\frac{1}{\alpha}}, \quad n \geq N \right. \right. \right. \]

Define the map \( T : \Omega \to B_N \) as follows:
\[ T_y = k - \frac{1}{2} \sum_{n=N_*}^{\infty} (s-n)q_r (y_{r(s)})^\beta \left| n \geq N \right. \]

The proof is similar to that of theorem 1 and there exists an element \( y \) such that \( y = T_y \), which is a type-\( I_2 \) solution of (1) with the property that \( \lim_{n \to \infty} \Delta y_n = k > 0 \), this completes the proof.
Theorem 6 The Equation (1) has a positive of type $\Pi_2$ if
\[
\sum_{n=1}^{\infty} \left( \sum_{n}^\infty (s-n)q_s(\tau(s))^\beta \right)^{\frac{1}{\alpha}} < \infty, \quad (22)
\]
and
\[
\sum_{n> n_0} \left( \sum_{n}^\infty (s-n)q_s(\tau(s))^\beta \right)^{\frac{1}{\alpha}} = \infty. \quad (23)
\]

Proof. Suppose now that (22) holds. Choose $N > n_0$ large enough so that
\[
\sum_{n=N}^{\infty} \left( \sum_{n}^\infty (s-n)q_s(\tau(s))^\beta \right)^{\frac{1}{\alpha}} \leq 2^{-\beta} k^{\frac{\beta}{\alpha}}. \quad (24)
\]
Put $N_* = \min \{N, \inf \tau(n)\}$. Let $B_N$ be the Banach space of all real sequences
\[
Y = \{y_n\}, \text{ with the norm } ||Y|| = \sup_{n>n_0} |y_n| < \infty,
\]
we define a closed, bounded and convex subset $\Omega$ of $B_N$ as follows:
\[
\Omega = \{ Y = \{y_n\} \in B_N : k \leq y_n \leq 2kn, \; n \geq N_* \}.
\]
Define the map $T : \Omega \rightarrow B_N$ as follows:
\[
T y_n = k + \sum_{s=n}^{\infty} \left( \sum_{s}^\infty (\sigma-r)q_s(\tau(s))^\beta \right)^{\frac{1}{\alpha}} n \geq N
\]
\[
T y_n = k \quad N_* \leq n < N
\]
(25)
The proof is similar to that of theorem 1 and there exists an element $y$ such that $y = Ty$, which is a type-$\Pi_2$ solution of (1). This completes the proof.

REFERENCES